Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

ON SOME COMPLETIONS OF THE SPACE OF HAMILTONIAN MAPS

Vincent Humilière

Tome 136
Fascicule 3

2008

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 373-404

Bull. Soc. math. France
136 (3), 2008, p. 373–404

ON SOME COMPLETIONS OF THE SPACE OF HAMILTONIAN MAPS

BY VINCENT HUMILIÈRE

ABSTRACT. — In one of his papers, C. Viterbo defined a distance on the set of Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ endowed with the standard symplectic form $\omega_0 = dp \wedge dq$. We study the completions of this space for the topology induced by Viterbo's distance and some others derived from it, we study their different inclusions and give some of their properties.

In particular, we give a convergence criterion for these distances that allows us to prove that the completions contain non-ordinary elements, as for example, discontinuous Hamiltonians. We also prove that some dynamical aspects of Hamiltonian systems are preserved in the completions.

RÉSUMÉ (*Sur certains complétés de l'espace des applications hamiltoniennes*)

Dans un de ses articles, C. Viterbo définit une distance sur l'ensemble des difféomorphismes hamiltoniens de $\mathbb{R}^{2n}$, muni de sa forme symplectique standard $\omega_0 = dp \wedge dq$. Nous étudions les complétés de cet espace pour la topologie induite par la distance de Viterbo, ainsi que d'autres qui en sont dérivées. Nous explicitons leurs différentes inclusions et donnons certaines de leur propriétés.

En particulier, nous donnons un critère de convergence pour ces distances qui nous permet de montrer que les complétés contiennent des éléments intéressants, comme, par exemple, des hamiltoniens discontinus. Nous prouvons aussi que certains aspects de la dynamique hamiltonienne sont préservés dans les complétés.

Texte reçu le 12 janvier 2007, révisé le 8 juin 2007, accepté le 11 janvier 2008

VINCENT HUMILIÈRE, Centre de Mathématiques Laurent Schwartz, UMR 7640 du CNRS, École polytechnique, 91128 Palaiseau, France • *E-mail :* `vincent.humiliere@math.polytechnique.fr`

2000 Mathematics Subject Classification. — 53D12, 37J05, 70H20.

Key words and phrases. — Symplectic topology, Hamiltonian dynamics, Viterbo distance, symplectic capacity, Hamilton-Jacobi equation.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2008/373/$ 5.00

1. Introduction

Given an open subset U in $\mathbb{R}^{2n}$, we denote by $\mathrm{Ham}(U)$ the set of all 1-periodic time dependent Hamiltonian functions $\mathbb{R}\times\mathbb{R}^{2n}\to\mathbb{R}$ whose support for fixed time is compact and contained in U. We will write Ham for $\mathrm{Ham}(\mathbb{R}^{2n})$.

Given a Hamiltonian function $H\in\mathrm{Ham}$, its symplectic gradient (i.e the unique vector field X_H satisfying $dH=\iota_{X_H}\omega_0$) generates a Hamiltonian isotopy $\{\phi_H^t\}$. The set of Hamiltonian diffeomorphisms generated by an element H in $\mathrm{Ham}(U)$ will be denoted by $\mathcal{H}(U)=\{\phi_H=\phi_H^1\,|\,H\in\mathrm{Ham}(U)\}$, and we will write $\mathcal{H}$ for $\mathcal{H}(\mathbb{R}^{2n})$. Finally, we call $\mathcal{L}=\{\phi(0_n)\,|\,\phi\in\mathcal{H}\}$, the set of Lagrangian submanifolds obtained from the zero section $0_n\subset T^*\mathbb{R}^n=\mathbb{R}^{2n}$, by a Hamiltonian isotopy with compact support.

As usual, we denote Viterbo's distance on $\mathcal{L}$ or $\mathcal{H}$ by γ (see [15]). Convergence with respect to γ is called c-convergence.

Our main goals in this paper is to understand the completion $\overline{\mathcal{H}}^\gamma$ of the metric space $(\mathcal{H},\gamma)$, to give some convergence criterion (section 3) and to compare it with the convergence for Hofer's distance d_H (see [5], chapter 5 section 1).

The notion of C^0 symplectic topology has been studied by many authors, starting from the work of Eliashberg and Gromov on the C^0-closure of the group of symplectic diffeomorphisms, to the later results of Viterbo ([15]) and Hofer ([4]).

More recently Oh ([9]) gave a deep study of several versions of C^0 Hamiltonians. However, our definition seems to differ from his, since in all his definitions, he needs the Hamiltonians to be continuous, while our study starts as we drop this assumption.

Let us now state our main results. For convenience, they will be restated throughout the paper. In section 3, we introduce a symplectic invariant ξ_∞ associated to any subset of $\mathbb{R}^{2n}$, and prove that

THEOREM 1.1. — *Let (H_k) be a sequence of Hamiltonians in* Ham*, whose supports are contained in a fixed compact set. Suppose there exist a Hamiltonian $H\in$* Ham *and a compact set $K\in\mathbb{R}^{2n}$ with $\xi_\infty(K)=0$, such that (H_k) converges uniformly to H on every compact set of $\mathbb{R}\times(\mathbb{R}^{2n}-K)$. Then (ϕ_{H_k}) converges to ϕ_H for γ.*

Examples of sets K with $\xi_\infty(K)=0$ are given by compact submanifolds of dimension $d\leqslant n-2$.

Viterbo's distance γ is defined on $\mathcal{H}$, but we can define for any $H,K\in\mathrm{Ham}$

$$\gamma_u(H,K)=\sup\{\gamma(\phi_H^t,\phi_K^t)\,|\,t\in[0,1]\},$$

to get a new distance on Ham (we give several variants of this definition). Then the following proposition allows to extend the notion of Hamiltonian flow.

PROPOSITION 1.2. — *If we consider the respective completions* $\overline{\mathcal{H}}^{\gamma}$ *and* $\overline{\mathrm{Ham}}^{\gamma_u}$ *of the metric spaces* $(\mathcal{H},\gamma)$ *and* (Ham,γ_u)*, then the map* $(H,t)\mapsto\phi_H^t$*,* $\mathrm{Ham}\times\mathbb{R}\to\mathcal{H}$ *induces a map* $\overline{\mathrm{Ham}}^{\gamma_u}\times\mathbb{R}\to\overline{\mathcal{H}}^{\gamma}$*.*

The induced map associates to any element H *in* $\overline{\mathrm{Ham}}^{\gamma_u}$ *a path in* $\overline{\mathcal{H}}^{\gamma}$ *that we will call the* generalized Hamiltonian flow *generated by* H*.*

We then show that some aspects of Hamiltonian dynamics can be extended to the completions (section 4): We can define a natural action of a generalized flow on a Lagrangian submanifold. We can also associate to it a support and extend the notion of first integral.

To some of them, it is also possible, as we prove in section 6, to associate a solution to the Hamilton-Jacobi equation:

$$\frac{\partial u}{\partial t}+H\left(t,x,\frac{\partial u}{\partial x}\right)=0.$$

Indeed, a γ_2-Cauchy sequence of Hamiltonians gives a C^0-Cauchy sequence of solutions (where γ_2 denotes one variant of the distance γ_u we mentioned above).

In section 5 we give examples of elements in both completions $\overline{\mathcal{H}}^{\gamma}$ and $\overline{\mathrm{Ham}}^{\gamma_u}$ that can be described in a much more concrete way than their abstract definition (as equivalence classes of Cauchy sequences). More precisely, we prove

PROPOSITION 1.3. — *There is a one-one map*

$$\mathfrak{F}^{\infty}\to\overline{\mathrm{Ham}}^{\gamma_u},$$

where $\mathfrak{F}^{\infty}$ *denotes the set of all functions* $H:\mathbb{R}\times\mathbb{R}^{2n}\to\mathbb{R}\cup\{+\infty\}$ *such that:*

(i) H *is continuous on* $\mathbb{R}\times\mathbb{R}^{2n}$*,*
(ii) H *vanishes at infinity:* $\forall\varepsilon>0,\exists r,(|x|>r\Rightarrow(\forall t,|H(t,x)|<\varepsilon))$*,*
(iii) *there exists a zero capacity set (e.g. an infinitesimally displaceable set), that contains all the points* x *where* $H(t,x)$ *is* $+\infty$ *for all time* t*,*
(iv) H *is smooth on* $\mathbb{R}\times\mathbb{R}^{2n}-H^{-1}(\{+\infty\})$*.*

Finally, let us mention that although we developed our theory on $\mathbb{R}^{2n}$, we can reasonably expect similar results (except those of sections 4.2 and 6) on any compact symplectic manifold satisfying

$$\omega|_{\pi_2(M)}=0 \text{ and } c_1|_{\pi_2(M)}=0.$$

Indeed, on these manifolds, Schwarz defined in [11] a distance which is entirely analogous to Viterbo's.

Organization of the paper. — In Section 2 we give the definitions of the objects used in the paper. For the reader's convenience, we first recall the construction of Viterbo's distance γ (2.2) which is based on the theory of generating functions for Lagrangian submanifolds (2.1). We also remind the reader of the different symplectic capacities constructed from γ (2.3). Finally we introduce our new distances derived from γ (2.4).

Section 3 is fully devoted to the proof of our convergence criterion. Examples of cases where it holds is then given in 3.3.

In Section 4 we define the completions of Ham and $\mathcal{H}$ and show that some aspects of Hamiltonian dynamics that can be extended to the completions.

In Section 5 we discuss some interesting examples of elements of the completions.

Our results on the Hamilton-Jacobi equation are given in Section 6.

Finally, we prove in Appendix a "reduction inequality" usefull to prove then all the inequalities between the distances considered in the paper.

Acknowledgments. — I am grateful to my supervisor C. Viterbo for his advices. I also want to thank my friends M. Affre and N. Roy for spending hours correcting my awful English.

2. Symplectic invariants

In this section we give the definitions of all the objects we will use in the sequel. We first recall the definition of Viterbo's distance, defined first for Lagrangian submanifolds with the help of generating functions, and then for Hamiltonian diffeomorphisms (see [15]).

2.1. Generating functions quadratic at infinity. — Let L be a Lagrangian submanifold of the cotangent bundle T^*M of a smooth manifold M. We say that L admits a *generating function* if there exists an integer $q > 0$ and a smooth function $S : M \times \mathbb{R}^q \to \mathbb{R}$ such that L can be written

$$L = \left\{(x,p) \in T^*M \mid \exists \xi \in \mathbb{R}^q, \frac{\partial S}{\partial \xi}(x,\xi) = 0 \text{ and } \frac{\partial S}{\partial x}(x,\xi) = p\right\}.$$

Such function S is called a *generating function quadratic at infinity* (or just "g.f.q.i") if there exists a non degenerate quadratic form Q on $\mathbb{R}^q$ and a compact $K \subset M \times \mathbb{R}^q$ such that, $\forall (x,\xi) \notin K, S(x,\xi) = Q(\xi)$.

For instance, any quadratic form on $\mathbb{R}^q$ viewed as a function on $M \times \mathbb{R}^q$ is a g.f.q.i of the zero section $0_M \subset T^*M$. J.C. Sikorav proved in [12] that the property of having a g.f.q.i is invariant by Hamiltonian isotopy with compact support. For this reason we will be interested in the set $\mathcal{L}$ of Lagrangian