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PARALLELEPIPEDS, NILPOTENT GROUPS AND GOWERS NORMS

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ABSTRACT. — In his proof of Szemerédi's Theorem, Gowers introduced certain norms that are defined on a parallelepiped structure. A natural question is on which sets a parallelepiped structure (and thus a Gowers norm) can be defined. We focus on dimensions 2 and 3 and show when this possible, and describe a correspondence between the parallelepiped structures and nilpotent groups.

RÉSUMÉ (*Parallélépipèdes, groupes nilpotents et normes de Gowers*)
Dans sa preuve du théorème de Szemerédi, Gowers a introduit certaines normes définies par sommation sur des parallélépipèdes. Il est naturel de se demander sous quelles hypothèses on peut généraliser sa définition des parallélépipèdes et donc de ses normes. Nous nous restreignons aux dimensions 2 et 3 et décrivons une correspondance entre structures de parallélépipèdes et groupes nilpotents.

1. Introduction

In his proof of Szemerédi's Theorem [17], Gowers [4] introduced certain norms for functions on $\mathbb{Z}/N\mathbb{Z}$. Shortly thereafter, these norms were adapted

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for a variety of other uses, including Green and Tao's major breakthrough showing that the primes contain arbitrarily long arithmetic progression [7] and their use in deriving finer asymptotics on structures in the primes ([5] and [6]). Similar seminorms were independently introduced by the authors and used to show convergence of some multiple ergodic averages [10]. Since then, the seminorms have been used for a variety of related problems in ergodic theory, including multiple averages along polynomial times ([9] and [15]), averages for certain commuting transformation [2] and averages along the primes [1].

Our goal here is to introduce and describe the most general context in which the first two Gowers norms can be defined. We call a 'parallelogram structure,' respectively a 'parallelepiped structure,' the weakest structure a set must have so that one can define a Gowers 2-norm, respectively a Gowers 3-norm, on the set.

The first Gowers norm is the absolute value of the sum of the values of the function, and in fact is only a seminorm. The second Gowers norm can be completely described using Fourier analysis (in terms of the ℓ^4 norm of the Fourier transform), and thus is closely linked to the abelian group structure of the circle. Analogously, in ergodic theory the second seminorm can be characterized completely by the Kronecker factor in a measure preserving system (see Furstenberg [3]), which is the largest abelian group rotation factor. The third Gowers norm is less well understood and can not be simply described in terms of Fourier analysis. In ergodic theory the third seminorm corresponds to a 2-step nilsystem, and more generally the k-th seminorm corresponds to a $(k-1)$-step nilsystem. (See [10] for the definition and precise statement; in the current context, the definition is given in Section 3.7.) In combinatorics, Green and Tao [5] have recently given a weak inverse theorem, but for the third Gowers norm the correspondence with a 2-step nilpotent group is not yet completely understood. We give conditions on a set that explain to what extent the correspondence with nilsystems can be made precise.

We start by defining the Gowers norms for $k \geq 2$. Let P denote the subset

$$\{(x_{00}, x_{01}, x_{10}, x_{11}) \in (\mathbb{Z}/N\mathbb{Z})^4 \colon x_{00} - x_{01} - x_{10} + x_{11} = 0\}$$

of $(\mathbb{Z}/N\mathbb{Z})^4$. For a function $f\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, the *second Gowers norm* $\|f\|_{U_2}$ is defined [(1)] by

$$\|f\|_{U_2}^4 = \sum_{(x_{00},x_{01},x_{10},x_{11})\in\mathsf{P}} f(x_{00})\overline{f(x_{01})}\,\overline{f(x_{10})}f(x_{11}) \ .$$

(Although this agrees with Gowers's original definition, Green and Tao prefer to normalize the sum and define the norm as an average instead of a sum. In

(1) The notation $\|\cdot\|_{U_k}$ was introduced later in the work of Green and Tao [7].

our context we prefer to work with the sum.) To define the norms U_k for $k \geq 3$, we need some notation.

Notation. — The elements of $\{0,1\}^k$ are written without commas and parentheses. For $\epsilon = \epsilon_1 \dots \epsilon_k \in \{0,1\}^k$ we write

$$|\epsilon| = \epsilon_1 + \cdots + \epsilon_k \ .$$

For $x = (x_1, \dots, x_k) \in (\mathbb{Z}/N\mathbb{Z})^k$ and $\epsilon \in \{0,1\}^k$, we write

$$\epsilon \cdot x = \epsilon_1 x_1 + \cdots + \epsilon_k x_k \ .$$

Let $C \colon \mathbb{C} \to \mathbb{C}$ denote complex conjugation. Therefore, for $n \in \mathbb{N} \cup \{0\}$ and $\xi \in \mathbb{C}$,

$$C^n \xi = \begin{cases} \xi & \text{if } n \text{ is even} \\ \overline{\xi} & \text{if } n \text{ is odd} \ . \end{cases}$$

Definition of the Gowers norms. — For $k \geq 3$, the k-th Gowers norm $\|f\|_{U_k}$ for a function $f \colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is defined to be the sum over k-dimensional parallelepipeds:

$$\|f\|_{U_k}^{2^k} = \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^k} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} \prod_{\epsilon \in \{0,1\}^k} C^{|\epsilon|} f(n + \epsilon \cdot x) \ .$$

For $k = 3$, $\|f\|_{U_3}^8$ can be written as the sum

$$\sum_{x,m,n,p \in \mathbb{Z}/N\mathbb{Z}} f(x)\overline{f(x+m)}\,\overline{f(x+n)}f(x+m+n) \\ \overline{f(x+p)}f(x+m+p)f(x+n+p)\overline{f(x+m+n+p)} \ . \tag{1}$$

A natural question is on which sets a parallelepiped structure, and thus a Gowers norm, can be defined. More interesting is understanding to what extent the correspondence with a k-step nilpotent group can be made in this more general setting. We restrict ourselves to the cases $k = 2$ and $k = 3$ and characterize to what extent this correspondence can be made precise. As the precise definitions of parallelogram and parallelepiped structures are postponed until we have developed some machinery, we only give a loose overview of the results. Essentially, the properties included in the definition of a parallelepiped structure are exactly those needed in order to define a Gowers type norm.

For a two dimensional parallelogram, we completely characterize possible parallelogram structures by an abelian group (Corollary 1). This means that a parallelogram structure arises from a 1-step nilpotent group. For the corresponding three dimensional case, the situation becomes more complex. In Theorem 1, Theorem 2 and Corollary 2 we show under some additional hypotheses, a parallelepiped structure corresponds to a 2-step nilpotent group.

However, there are examples (Example 6) for which this hypothesis is not satisfied. On the other hand, we are able to show (Theorem 3) that in all cases, the parallelepiped structure can be embedded in a 2-step nilsystem.

In Section 7, we outline to what extent these results can be carried out in higher dimensions.

For all sets on which it is possible to define these structures, one can naturally define the corresponding Gowers norm U_k. We expect that these norms should have other applications, outside of those already developed by Gowers, Green and Tao, and the authors. The results of this paper already have an application in topological dynamics by Host and Maass [11], where they give a new characterization of 2-step nilsystems and 2-step nilsequences.

More notation. — Parallelogram structures and parallelepiped structures are defined as subsets of the Cartesian powers X^4 and X^8 of some sets or groups and so we introduce some notation.

When X is a set, we write $X^{[2]} = X \times X \times X \times X$ and let $X^{[3]}$ denote the analogous product with 8 terms.

A point in $X^{[2]}$ is written $\mathbf{x} = (x_0, x_1, x_2, x_3)$ or $\mathbf{x} = (x_{00}, x_{01}, x_{10}, x_{11})$ and a point in $X^{[3]}$ is written $\underline{\mathbf{x}} = (x_0, \dots, x_7)$ or $\underline{\mathbf{x}} = (x_{000}, x_{001}, \dots, x_{111})$. More succinctly, we denote $\mathbf{x} \in X^{[2]}$ by $\mathbf{x} = (x_i \colon 0 \leq i \leq 3)$ or $\mathbf{x} = (x_\epsilon \colon \epsilon \in \{0,1\}^2)$, and use similar notation for points in $X^{[3]}$.

It is convenient to identify $\{0,1\}^2$ with the set of vertices of the Euclidean unit square. Then the second type of notation allows us to view each coordinate of a point $\mathbf{x}$ of $X^{[2]}$ as lying at the corresponding vertex.

Each Euclidean isometry of the square permutes the vertices and thus the coordinates of $\mathbf{x}$. The permutations of $X^{[2]}$ defined in this way are called the *Euclidean permutations of* $X^{[2]}$. For example, the maps

$$\mathbf{x} \mapsto (x_{10}, x_{11}, x_{00}, x_{01}) \text{ and } \mathbf{x} \mapsto (x_{10}, x_{00}, x_{11}, x_{01})$$

are Euclidean permutations. We use the same vocabulary for $X^{[3]}$, with the Euclidean 3-dimensional unit cube replacing the square.

If $r\colon X \to Y$ is a map, by $r^{[2]}\colon X^{[2]} \to Y^{[2]}$ we mean

$$r^{[2]}(\mathbf{x}) = \big(r(x_{00}), r(x_{01}), r(x_{10}), r(x_{11})\big) \ .$$

Similarly, $r^{[3]}$ is defined as the corresponding map $r^{[3]}\colon X^{[3]} \to Y^{[3]}$.