Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

CONVOLUTION SQUARE, SELF CONVOLUTION, SINGULAR MEASURE

Thomas Körner

Tome 136
Fascicule 3

2008

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 439-464

Bull. Soc. math. France
136 (3), 2007, p. 439–464

ON A THEOREM OF SAEKI CONCERNING CONVOLUTION SQUARES OF SINGULAR MEASURES

BY THOMAS KÖRNER

ABSTRACT. — If $1 > \alpha > 1/2$, then there exists a probability measure μ such that the Hausdorff dimension of the support of μ is α and $\mu * \mu$ is a Lipschitz function of class $\alpha - \frac{1}{2}$.

RÉSUMÉ (*Carrés de convolution des mesures singulières*). — Si $1 > \alpha > 1/2$, alors il existe une mesure de probabilité μ avec support de dimension d'Hausdorff α tel que $\mu * \mu$ est une fonction Lipschitz de classe $\alpha - \frac{1}{2}$.

1. Introduction

We work on on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We write τ for the Lebesgue measure on $\mathbb{T}$ and $|E| = \tau(E)$. We say that a function $f : \mathbb{T} \to \mathbb{C}$ is Lipschitz β if

$$\sup_{t\in\mathbb{T}} \sup_{h\neq 0} |h|^{-\beta}|f(t+h) - f(t)| < \infty$$

for some $1 \geq \beta > 0$.

In a famous paper [11], Wiener and Wintner constructed a singular measure μ such that $\mu * \mu \in L^p(T)$ for all $p \geq 1$ and other authors have given further examples along these lines (see Chapter 6 of [2] and [4]). The strongest result is due to Saeki [10] who constructs a singular measure μ with support of Lebesgue

Texte reçu le 22 février 2007, révisé le 6 novembre 2007

THOMAS KÖRNER, DPMMS, Centre for Mathematical Sciences, Clarkson Road, Cambridge
• *E-mail :* `twk@dpmms.cam.ac.uk`

2000 Mathematics Subject Classification. — Primary: 42A16.

Key words and phrases. — Convolution square, self convolution, singular measure.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2007/439/$ 5.00

measure zero such that $\mu * \mu = f\tau$ where f has a uniformly convergent Fourier series. He also remarks that this can be improved to give $\mu * \mu = f\tau$ with f Lipschitz β provided that $\beta < 1/2$, but leaves the proof as an exercise to the reader. This paper may be considered as an extension to that exercise, although I believe that the results obtained are more precise and that the method used is not that envisaged by Saeki.

The object of this paper is to prove the following theorem.

THEOREM 1. — *If $1 > \alpha > 1/2$, then there exists a probability measure μ such that the Hausdorff dimension of the support of μ is α and $\mu * \mu = f\tau$ where f is Lipschitz $\alpha - \frac{1}{2}$.*

At the two extremes $\alpha = 1$ and $\alpha = 1/2$ we get the following versions of Theorem 1.

THEOREM 2. — *There exists a probability measure μ such that the support of μ has Lebesgue measure 0 and $\mu * \mu = f\tau$ where f is Lipschitz β for all $\beta < 1/2$.*

THEOREM 3. — *There exists a probability measure μ such that the Hausdorff dimension of the support of μ is $1/2$ and $\mu * \mu = f\tau$ where f is continuous with uniformly convergent Fourier series.*

We shall see, in Corollary 23, how to show that all these results remain true if we replace $\mathbb{T}$ by $\mathbb{R}$. We note a consequence of Corollary 23 here.

LEMMA 4. — *Suppose $G : \mathbb{R} \to \mathbb{R}$ is a positive continuous function of bounded support. Then, given any $\epsilon > 0$, we can find a positive measure σ with support of Lebesgue measure zero such that $\sigma * \sigma = F\tau_{\mathbb{R}}$ where $\tau_{\mathbb{R}}$ is Lebesgue measure and F is continuous with $\|F - G * G\|_\infty < \epsilon$.*

This indicates that the any numerical method for finding the approximate 'convolution square root' must overcome substantial difficulties.

The next lemma shows that, at the relatively coarse level of Hausdorff dimension and Lipschitz coefficients, these results must be close to best possible.

LEMMA 5. — (i) *If μ is a measure whose support has Hausdorff dimension α and $\mu * \mu = f\tau$ where f is Lipschitz β, then $\alpha - \frac{1}{2} \geq \beta$.*

(ii) *If μ is a measure whose support has Hausdorff dimension α and $\mu * \mu = f\tau$ where f is continuous, then $\alpha \geq \frac{1}{2}$.*

Proof. — (i) Since f is Lipschitz β, it follows that (see, for example, [1] Chapter II §3)

$$\sum_{n\leq|k|\leq 2n-1} |\hat{f}(k)| \leq C_1 n^{(1-2\beta)/2}$$

for some constant C_1 depending on f. Since $|\hat{f}(k)| = |\hat{\mu}(k)|^2$, we have

$$\sum_{n\leq|k|\leq 2n-1} |\hat{\mu}(k)|^2 \leq C_1 n^{(1-2\beta)/2}$$

and so, if $\eta > 0$,

$$\sum_{k=n}^{2n-1} \frac{|\hat{\mu}(k)|^2}{|k|^{1-\eta}} \leq C_2 n^{-(1+2\beta-2\eta)/2}$$

for all $n \geq 1$ and some constant C_2. By Cauchy's condensation test,

$$\sum_{k\neq 0} \frac{|\hat{\mu}(k)|^2}{|k|^{1-\eta}} \text{ converges}$$

whenever $(1+2\beta)/2 > \eta$.

We know (by Theorems I and V of Chapter 3 in [7]), that if σ is a non-zero measure with

$$\sum_{k\neq 0} \frac{|\hat{\sigma}(k)|^2}{|k|^{1-\eta}} \text{ convergent}$$

for some $0 < \eta < 1$, it follows that the Hausdorff dimension of $\operatorname{supp}\mu$ must be at least η. Thus the Hausdorff dimension of $\operatorname{supp}\mu$ must be at least η for each η with $(1+2\beta)/2 > \eta$. We conclude that the Hausdorff dimension of $\operatorname{supp}\mu$ must be at least $(1+2\beta)/2$.

(ii) This follows the proof of (i) with $\beta = 0$. □

Our method of proof gives two slightly stronger versions of the theorems announced above which we state as Theorems 7 and 8. We need a preliminary pair of definitions.

Definition 6. — *Suppose that $\psi : \mathbb{R} \to \mathbb{R}^+$ is a strictly increasing continuous function with $\psi(0) = 0$.*

(i) *We say that a set $E \subseteq \mathbb{T}$ has Hausdorff ψ measure zero if, given $\epsilon > 0$, we can find a countable collection $\mathcal{I}$ of closed intervals such that*

$$\bigcup_{I\in\mathcal{I}} I \supseteq E \text{ and } \sum_{I\in\mathcal{I}} \psi(|I|) < \epsilon.$$

(ii) *We say that function $f : \mathbb{T} \to \mathbb{C}$ lies in Λ_ψ if*

$$\sup_{t,h\in\mathbb{T}, h\neq 0} \psi(|h|)^{-1} |f(t+h) - f(t)| < \infty.$$

THEOREM 7. — *Let $1 > \alpha \geq 1/2$. Suppose that $\psi : \mathbb{R} \to \mathbb{R}^+$ is a strictly increasing continuous function with $\psi(0) = 0$ and $t^{-\beta}\psi(t) \to 0$ as $t \to 0+$ whenever $\beta < \alpha - \frac{1}{2}$. Then there exists a probability measure μ such that the Hausdorff dimension of the support of μ is α and $\mu * \mu = f\tau$ with $f \in \Lambda_\psi$.*

If $1 > \alpha > 1/2$ and $\psi(t) = t^{\alpha-1/2}$, we recover Theorem 1. If $\alpha = 1/2$ and $\psi(t) = (\log t^{-1})^{-2}$ for $0 < t < 1/2$, we recover Theorem 3 since the Dini–Lipschitz test tells us that any function in Λ_ψ must have uniformly convergent Fourier series. (See, for example, [1] Chapter IV §4.)

THEOREM 8. — *Let $1 \geq \alpha > 1/2$. Suppose that $\psi : \mathbb{R} \to \mathbb{R}^+$ is a strictly increasing continuous function with $\psi(0) = 0$ and $t^{-\beta}\psi(t) \to 0$ as $t \to 0+$ whenever $\beta < \alpha - \frac{1}{2}$. Then there exists a probability measure μ such that* supp μ *has Hausdorff ψ measure zero and $\mu * \mu = f\tau$ with f Lipschitz β for all $\beta < \alpha - \frac{1}{2}$.*

If $\alpha = 1$ and $\psi(t) = t$, we recover Theorem 2.

Section 2 is devoted to the proof of the key Lemma 9. The proof is probabilistic and, as in several other papers, I acknowledge the influence of Kaufman's elegant note [8]. In the Section 3 we smooth the result of Lemma 9 to obtain Lemma 17 which we later use in Section 5 to prove a Baire category version of Theorem 7. In the final section we sketch the very similar proof of a Baire category version of Theorem 8.

In my opinion, the main ideas of the paper are to be found in the proofs of Lemmas 10 and 26.

2. The basic construction

The key to our construction is the following lemma.

LEMMA 9. — *Suppose $\phi : \mathbb{N} \to \mathbb{R}$ is a sequence with $\phi(n) \to \infty$ as $n \to \infty$. If $1 > \gamma > 0$ and $\epsilon > 0$, there exist an $M(\gamma)$ and $n_0(\phi, \gamma, \epsilon) \geq 1$ with the following property. If $n \geq n_0(\phi, \gamma, \epsilon)$, n is odd and $n^\gamma \geq N$ we can find N points*

$$x_j \in \{r/n : r \in \mathbb{Z}\}$$

(not necessarily distinct) such that, writing

$$\mu = N^{-1} \sum_{j=1}^{N} \delta_{x_j},$$

we have

$$|\mu * \mu(\{k/n\}) - n^{-1}| \leq \epsilon \frac{\phi(n)(\log n)^{1/2}}{N n^{1/2}}$$