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INTEGRAL REPRESENTATIONS FOR SOLUTIONS OF EXPONENTIAL GAUSS-MANIN SYSTEMS

BY MARCO HIEN & CÉLINE ROUCAIROL

ABSTRACT. — Let $f, g : U \to \mathbb{A}^1$ be two regular functions from the smooth affine complex variety U to the affine line. The associated exponential Gauß-Manin systems on the affine line are defined to be the cohomology sheaves of the direct image of the exponential differential system $\mathcal{O}_U e^g$ with respect to f. We prove that its holomorphic solutions admit representations in terms of period integrals over topological chains with possibly closed support and with rapid decay condition.

RÉSUMÉ (*Représentations intégrales des solutions des systèmes de Gauß-Manin exponentiels*)

Soient $f, g : U \to \mathbb{A}^1$ deux fonctions régulières sur une variété affine lisse U à valeurs dans la droite affine. On leurs associe des systèmes de Gauß-Manin sur la droite affine définis comme étant les faisceaux de cohomology de l'image directe par f du système différentiel exponentiel $\mathcal{O}_U e^g$. Nous prouvons que leurs solutions holomorphes admettent des représentations sous forme d'intégrales de périodes sur des chaînes topologiques à support éventuellement fermé avec une condition de décroissance rapide.

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1. Introduction

Let U be a smooth affine complex variety of dimension n and let $f, g : U \to \mathbb{A}^1$ be two regular functions. We consider the flat algebraic connection $\nabla : \mathcal{O}_U \to \Omega^1_U$ on the trivial line bundle $\mathcal{O}_U$ defined as $\nabla u = du + dg \cdot u$. Let us denote the associated holonomic $\mathcal{D}_U$-module by $\mathcal{O}_U e^g$, i.e. the trivial $\mathcal{O}_U$-module on which any vector field ξ on U acts via the associated derivation ∇_ξ induced by the connection. Let $\mathcal{N}$ be the direct image $\mathcal{N} := f_+(\mathcal{O}_U e^g)$ in the theory of $\mathcal{D}$-modules (see e.g. [2]). Then $\mathcal{N}$ is a complex of $\mathcal{D}_{\mathbb{A}^1}$-modules and we consider its k-th cohomology sheaf

$$\mathcal{H}^k f_+(\mathcal{O}_U e^g),$$

a holonomic $\mathcal{D}_{\mathbb{A}^1}$-module.

There is an alternative point of view of $\mathcal{H}^k\mathcal{N}$ in terms of flat connections: outside a finite subset $\Sigma_1 \subset \mathbb{A}^1$, the $\mathcal{O}_{\mathbb{A}^1}$-module $\mathcal{H}^k f_+(\mathcal{O}_U e^g)$ is locally free, hence a flat connection. This connections coincides with the Gauß-Manin connection on the relative de Rham cohomology

$$\mathcal{H}^k(\mathcal{N})|_{\mathbb{A}^1 \smallsetminus \Sigma_1} \cong \big(\boldsymbol{R}^{k+n-1} f_*(\Omega^\bullet_{U/\mathbb{A}^1}, \nabla), \nabla_{GM}\big)|_{\mathbb{A}^1 \smallsetminus \Sigma_1},$$

i.e. the corresponding higher direct image of the complex of relative differential forms on U with respect to f and the differential induced by the absolute connection, as in [6]. It is a flat connection over $\mathbb{A}^1 \smallsetminus \Sigma_1$.

In the case $g = 0$, which is classically called the Gauß-Manin system of f, it is well-known that the solutions admit integral representations (see [10] and [12]). The aim of the present article is to give such a description of the solutions in the more general case of the Gauß-Manin system of the exponential module $\mathcal{O}_U e^g$, which we will call *exponential Gauß-Manin system*. A major difficulty lies in the definition of the integration involved. In the case of vanishing g, the connection is regular singular at infinity and the topological cycles over which the integrations are performed can be chosen with compact support inside the affine variety U. In the exponential case $g \neq 0$, the connection is irregular singular at infinity and we will have to consider integration paths approaching the irregular locus at infinity.

A systematic examination of period integrals in more general irregular singular situations over complex surfaces has been carried out in [5] resulting in a perfect duality between the de Rham cohomology and some homology groups, the *rapid decay homology groups*, in terms of period integrals. We will prove in this article that this result generalizes to arbitrary dimension in case of an elementary exponential connection $\mathcal{O}e^g$ as before, see Theorem 2.7.

With this tool at hand, we construct a local system $\mathfrak{H}^{rd}_k$ on $\mathbb{A}^1 \smallsetminus \Sigma_2$, the stalk at point a t of which is the rapid decay homology $H^{rd}_k(f^{-1}(t), e^{-g_t})$ of the restriction of $\mathcal{O}_U e^{-g}$ to the fibre $f^{-1}(t)$. Let $\Sigma = \Sigma_1 \cup \Sigma_2$. We prove that given a section $c_t \otimes e^{g_t}$ in the local system $\mathfrak{H}^{rd}_{k+n-1}$ and a relative differential form ω in $\Omega^{k+n-1}_{rel}(f^{-1}(\mathbb{A}^1 \smallsetminus \Sigma))$ (which describes the analytification of $\mathcal{H}^k\mathcal{N}$ as in the obvious analytic analogue to Proposition 3.3), the integral

$$\int_{c_t} \omega|_{f^{-1}(t)} \cdot e^{g_t} \tag{1}$$

gives a (multi-valued) holomorphic solution of the exponential Gauß-Manin system. As a consequence of the duality proved in Theorem 2.7, we deduce that all (multi-valued) holomorphic solutions on $\mathbb{A}^1 \smallsetminus \Sigma$ can be obtained by this construction, i.e. we achieve the following theorem (Theorem 3.5) which is the main result of this article:

THEOREM 1.1. — *For any simply connected open subset $V \subset \mathbb{C} \smallsetminus \Sigma$, the space of holomorphic solutions of the exponential Gauß-Manin system $\mathcal{H}^k\mathcal{N}$ is isomorphic to the space of sections in the local system $\mathfrak{H}^{rd}_{k+n-1}$ over V.*

The isomorphism in the theorem is given in terms of the integration (1). We want to remark that this theorem generalizes similar results by F. Pham for the Fourier-Laplace transform of the Gauß-Manin systems of a regular function $h : U \to \mathbb{A}^1$ (consider $f : \mathbb{A}^1 \times U \to \mathbb{A}^1$ to be the first canonical projection and $g : \mathbb{A}^1 \times U \to \mathbb{A}^1$ given by $g(x,u) = xh(u)$, cf. [11]). It also includes other well-known examples as the integral representations of the Bessel-functions for instance (cf. the introduction of [1] and the final remark of this article).

2. The period pairing

Consider the situation described in the introduction, namely $f, g : U \to \mathbb{A}^1$ being two regular functions on the smooth affine complex variety U. The main result of this work is to give a representation of the Gauß-Manin solutions in terms of period integrals. The proof relies on a duality statement between the algebraic de Rham cohomology of the exponential connection associated to g and some Betti homology groups with decay condition. The present section is devoted to the proof of this duality statement, which is a generalization of the analogous result for surfaces in [5] to the case of exponential line bundles in arbitrary dimensions.

2.1. Rapid decay homology. — Let us start with any smooth affine complex variety U_t and a regular function $g_t : U_t \to \mathbb{A}^1$. Note, that the index t is irrelevant in this section but will become meaningful later in the application to the Gauß-Manin system where U_t will denote the smooth fibres $f^{-1}(t)$.

We assume that U_t is embedded into a smooth projective variety X_t and that g_t extends to a meromorphic mapping $g_t : X_t \to \mathbb{P}^1$. Note, that we do not impose any further conditions on the pole divisor $D_t := g_t^{-1}(\infty)$.

Let us denote by $\mathcal{O}_{X_t}[*D_t]$ the sheaf of meromorphic functions on X_t with poles along D_t. We will write $\mathcal{O}e^{g_t}$ for the flat meromorphic connection

$$\nabla : \mathcal{O}_{X_t}[*D_t] \longrightarrow \Omega^1_{X_t/\mathbb{C}}[*D_t],\ u \mapsto du + u \cdot dg_t$$

on the trivial meromorphic line bundle $\mathcal{O}_{X_t}[*D_t]$.

In [5], a duality pairing between the de Rham cohomology of a flat meromorphic connection with poles along a divisor with normal crossings (admitting a good formal structure) and a certain homology theory, the rapid decay homology, is constructed in the case $\dim(X_t) = 2$, generalizing previous constructions of Bloch and Esnault for curves. For connections of exponential type lying in the main focus of this work, we will now generalized these results to the case of arbitrary dimension and also weakening the normal crossing assumption on the pole divisor. We remark that a crucial point in [5] is to work with a *good* compactification with respect to the given vector bundle and connection. For connections of the type $\mathcal{O}e^{g_t}$ as above, we can always pull-back to such a good situation by a finite blow-up process.

We now want to define the rapid decay homology of the dual connection $\mathcal{O}e^{-g_t}$. Since we do not assume that D_t is a normal crossing divisor, we have to modify the definition of [5] for our present purposes. We start with the given meromorphic map $g_t : X_t \to \mathbb{P}^1$. Consider the real oriented blow-up $\pi : \widetilde{\mathbb{P}^1} \to \mathbb{P}^1$ of the point ∞ in $\mathbb{P}^1$ and let $S^1_\infty := \pi^{-1}(\infty)$ be the circle at infinity. We will use the following local presentation of the real oriented blow-up of $\infty \in \mathbb{P}^1$

$$\pi^{-1}(\mathbb{P}^1 \smallsetminus 0) \cong \mathbb{R}^+_0 \times S^1 \xrightarrow{\pi} \mathbb{P}^1 \smallsetminus 0,\ (r, e^{i\vartheta}) \mapsto \frac{1}{r} e^{i\vartheta}$$

and identify $S^1_\infty = \pi^{-1}(\infty)$ with $0 \times S^1$. We define

$$\widetilde{X_t} = X_t \times_{g_t} \widetilde{\mathbb{P}^1} \tag{2}$$

as the fibre product of X_t and $\widetilde{\mathbb{P}^1}$ over $\mathbb{P}^1$ and denote the associated maps as in the following diagram:

$$\begin{array}{ccc} \widetilde{X_t} & \xrightarrow{\widetilde{g_t}} & \widetilde{\mathbb{P}^1} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi} \\ X_t & \xrightarrow{g_t} & \mathbb{P}^1 \end{array} \tag{3}$$