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LINEARIZATION OF GERMS: REGULAR DEPENDENCE ON THE MULTIPLIER

by Stefano Marmi & Carlo Carminati

Abstract. — We prove that the linearization of a germ of holomorphic map of the type $F_\lambda(z) = \lambda(z + O(z^2))$ has a $\mathcal{C}^1$-holomorphic dependence on the multiplier λ. $\mathcal{C}^1$-holomorphic functions are $\mathcal{C}^1$-Whitney smooth functions, defined on compact subsets and which belong to the kernel of the $\bar{\partial}$ operator.

The linearization is analytic for $|\lambda| \neq 1$ and the unit circle $\mathbb{S}^1$ appears as a natural boundary (because of resonances, *i.e.* roots of unity). However the linearization is still defined at most points of $\mathbb{S}^1$, namely those points which lie "far enough from resonances", i.e. when the multiplier satisfies a suitable arithmetical condition. We construct an increasing sequence of compacts which avoid resonances and prove that the linearization belongs to the associated spaces of $\mathcal{C}^1$-holomorphic functions. This is a special case of Borel's theory of uniform monogenic functions [2], and the corresponding function space is arcwise-quasianalytic [11]. Among the consequences of these results, we can prove that the linearization admits an asymptotic expansion w.r.t. the multiplier at all points of the unit circle verifying the Brjuno condition: in fact the asymptotic expansion is of Gevrey type at diophantine points.

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Résumé (*Linéarisation des germes : dépendence régulière du multiplicateur*)

On montre que la linéarisation d'un germe d'application holomorphe du type $F_\lambda(z) = \lambda(z + O(z^2))$ a une dépendence $\mathcal{C}^1$-holomorphe du multiplicateur λ. Les fonctions $\mathcal{C}^1$-holomorphes sont $\mathcal{C}^1$ au sens de Whitney, elles sont définies sur des compacts et elles appartiennent au noyau de l'operateur $\bar{\partial}$.

La linéarisation est analytique pour $|\lambda| \neq 1$ et le circle $\mathbb{S}^1$ est sa frontière naturelle (due aux résonances, c'est-à-dire les racines de l'unité). Neamoins la linéarisation est encore définie dans la plupart des points de $\mathbb{S}^1$, plus précisement aux points qui se trouvent « assez loin des résonances »' et qui correspondent à des conditions arithmétiques adéquates imposées au multiplicateur. Nous construisons une suite croissante d'ensembles compacts qui évitent les résonances et nous démontrons que la linéarisation appartient aux espaces associés aux fonctions $\mathcal{C}^1$-holomorphes. C'est un cas particulier de la théorie des fonctions monogènes uniformes de Borel [2], et les espaces de fonctions correspondants sont quasi-analytiques par chemins [11]. Comme conséquence nous montrons que la linéarisation a un développement asymptotique en $(\lambda - \lambda_0)$ dans tous les points $\lambda_0 \in \mathbb{S}^1$ qui verifient la condition de Brjuno. En effet le developpement est du type Gevrey aux points diophantiens.

1. Introduction

A germ of holomorphic diffeomorphism of $(\mathbb{C}, 0)$

$$F_\lambda(z) = \lambda(z + \sum_{k=2}^{+\infty} f_k z^k), \qquad (\lambda \in \mathbb{C}^*) \tag{1}$$

is *linearizable* if there exists a holomorphic germ tangent to the identity $H_\lambda(z) = z + \sum_2^{+\infty} h_k(\lambda) z^k$ which conjugates F_λ to the rotation $R_\lambda : z \mapsto \lambda z$ namely

$$F_\lambda \circ H_\lambda = H_\lambda \circ R_\lambda. \tag{2}$$

The derivative λ of F_λ at the fixed point $z = 0$ is called the multiplier of F_λ.

If λ is not a root of unity there exists a unique formal solution to the conjugacy equation with coefficients h_k, $k \geq 2$, determined by the recurrence relation

$$h_k = \frac{1}{\lambda^{k-1} - 1} \sum_{j=2}^{k} f_j \sum_{\substack{\epsilon \in (\mathbb{Z}_+)^j \\ |\epsilon| = k}} h_{\epsilon_1} \cdots h_{\epsilon_j}, \tag{3}$$

where we follow the usual multi-index notation $|\epsilon| = \sum_{i=1}^{j} \epsilon_i$. Note that $h_k \in \mathbb{C}(\lambda)[f_2, \ldots, f_k]$. When $|\lambda| \neq 1$ F_λ is always linearizable (by the classical Koenigs-Poincaré theorem); nevertheless the classical estimates on radius of convergence of H_λ deteriorate as $|\lambda| \to 1$. In the elliptic case, i.e. when $\lambda = e^{2\pi i \alpha}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the linearization need not be convergent due to the contribution from small denominators in (3). After the work of Brjuno [3] and Yoccoz [18] we know that all holomorphic germs with multiplier

$\lambda = e^{2\pi i\alpha}$ are analytically linearizable if and only if α verifies the Brjuno condition $\mathcal{B}(\alpha) < +\infty$, where $\mathcal{B}$ is the Brjuno function (see the next Section for its definition and properties).

Let us normalize F_λ asking that it is defined and univalent on the unit disk $\mathbb{D}$. Then one can prove directly, using the classical majorant series' method and Davie's Lemma (see [5] [4]), that there are positive constants b_0, c_0 (that do not depend on α) such that

$$|h_k| \le c_0 e^{k(\mathcal{B}(\alpha)+b_0)} \tag{4}$$

where $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R}$ and $\mathcal{B}$ is the Brjuno function [(1)].

The same estimate (4) (with larger values b_0 and c_0) holds uniformly with respect to λ' in a cone with vertex in $e^{2\pi i\alpha}$. Therefore for any $\varepsilon > 0$ we will be able to define a closed set C such that there exists $\rho > 0$ such that

- (i) $\mathrm{meas}_2(\mathbb{C} \setminus C) \le \varepsilon$ and $\mathrm{meas}_1(C \cap \mathbb{S}^1) \ge 2\pi - \varepsilon$,
- (ii) for each $\lambda \in C$ the linearization H_λ is holomorphic and bounded on $\mathbb{D}_\rho = \{z \in \mathbb{C} \ : \ |z| \le \rho\}$.

(Here meas_d, $d \in \{1, 2\}$ denotes the d-dimensional Lebesgue measure).

The construction of such a set is performed removing from $\mathbb{C}$ the union of suitably small connected open neighbourhoods around the roots of unity [(2)] and its detailed description can be found in Section 3.1; it will be evident from the construction that the radius ρ tends to 0 as ϵ tends to 0.

Let us point out that the property (ii) above means that a uniform lower bound on the radius of convergence of H_λ holds as λ varies in C, even near the unit circle. The set C is sort of a "bridge" joining the two connected components of the set of parameter values considered in the Koenigs-Poincaré theorem, crossing the unit circle at some values $\lambda = e^{2\pi i\alpha}$ with α a Brjuno number.

We address the problem of studying the regularity of this map $\lambda \mapsto H_\lambda$: we will prove *global regularity results* (see Theorem A below) and *local regularity results* (Theorem B).

The global regularity results we prove are inspired by the work of Borel on uniform monogenic functions [2]. Borel extended the notion of holomorphic function so as to include functions defined on closed subsets of $\mathbb{C}$. His uniform monogenic functions (whose precise definition we recall and recast in modern terminology in Appendix B) can have, in certain situations, analytic

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(1) It is known that there are different objects that are called "Brjuno function"; nevertheless for this estimate is quite irrelevant which one we choose, since the difference of two Brjuno functions is bounded by a universal constant, i.e. independent of α (see Section 2).

(2) The property (i) can be realized just asking that the "size" of the neighbourhood of each root decays suffcently fast when the order of the root increases.

continuation through what is considered as a natural boundary of analyticity in Weierstrass' theory. One of Borel's goals was to determine, with the help of Cauchy's formula, sufficiently general conditions which would have ensured uniqueness of the monogenic continuation, *i.e.* a quasianalyticity property (see [15], [17] for a modern discussion of this part).

The importance of Borel's monogenic functions in parameter-dependent small divisor problems was emphasized by Kolmogorov [9]. Arnol'd discussed in detail this issue in his work [1] on the local linearization problem of analytic diffeomorphisms of the circle (see [19] for a very nice introduction and for the most complete results on the subject). Arnol'd complexified the rotation number but he did not prove that the dependence of the conjugacy on it is monogenic. This point was dealt with by M. Herman [7] who also reformulated Borel's ideas using the modern terminology, Whitney's theory [16] on differentiability of functions on closed sets and the theory of uniform algebras of (analytic) functions defined on closed sets in the complex plane. It is Herman's point of view which was developed in [10] and which we will summarize in Appendix B, where we recall the formal definition of $\mathcal{C}^1$-holomorphic and $\mathcal{C}^\infty$-holomorphic functions. Later Risler [13] extended considerably part of Herman's work proving various regularity results under less restrictive arithmetical conditions, namely using the Brjuno conditon as in [19] instead of a more classical diophantine condition. One should also mention that Whitney smooth dependence on parameters has been established also in the more general framework of KAM theory by Pöschel [12] who did not however consider neither complex frequencies nor Brjuno numbers.

In this paper we will extend the results of Herman and Riesler to the case of germs of holomorphic diffeomorphisms of $(\mathbb{C},0)$. Our proofs will in fact be more elementary since in this case one can use a direct approach and the majorant series method applies (see, e.g. [4]).

Let us point out that, although the linearization problem makes no sense for $\lambda = 0$, nevertheless the recurrence (3) defines a function $H : \lambda \mapsto H_\lambda$ which turns out to be well defined and holomorphic at the origin: if we denote with $\mathcal{F}_0 = z + \sum_{k=2}^{+\infty} f_k z^k$, so that $F_\lambda = \lambda \mathcal{F}_0$, then H_0 turns out to be simply the inverse of $\mathcal{F}_0$: $\mathcal{F}_0(H_0(z)) = z$. In fact H can even be extended analitically at infinity just setting $H_\infty(z) = z$. Therefore we may consider H as defined on $C \cup \{\infty\}$ which is a compact subset of $\mathbb{P}^1\mathbb{C}$: this has an important consequence since it is proved in [11] that the space of $\mathcal{C}^1$-holomorphic functions to which H belongs (see Theorem A below) is *arcwise quasianalytic* [(3)].

(3) A function space X is said to be *arcwise quasianalytic* iff the only function that belongs to X and vanishes on an arbitrarily short arc is the null function.