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TRIVIALIZATION OF $\mathcal{C}(X)$-ALGEBRAS WITH STRONGLY SELF-ABSORBING FIBRES

by Marius Dadarlat & Wilhelm Winter

Abstract. — Suppose A is a separable unital $\mathcal{C}(X)$-algebra each fibre of which is isomorphic to the same strongly self-absorbing and K_1-injective C^*-algebra $\mathcal{D}$. We show that A and $\mathcal{C}(X) \otimes \mathcal{D}$ are isomorphic as $\mathcal{C}(X)$-algebras provided the compact Hausdorff space X is finite-dimensional. This statement is known not to extend to the infinite-dimensional case.

Résumé (*Trivialisation de $\mathcal{C}(X)$-algèbres à fibres fortement auto-absorbantes*)

Soit A une $\mathcal{C}(X)$-algèbre séparable unital dont chaque fibre est isomorphe à une même C^*-algèbre $\mathcal{D}$ K_1-injective et fortement auto-absorbante. Nous montrons que si l'espace compact et Hausdorff X est de dimension finie, alors A et $\mathcal{C}(X) \otimes \mathcal{D}$ sont isomorphes en tant que $\mathcal{C}(X)$-algèbres. Ce resultat est connu pour ne pas s'étendre au cas des espaces de dimension infinie.

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1. Introduction

A classical problem in the theory of C^*-algebras is to classify C^*-algebras with primitive spectrum a given compact Hausdorff space X. These algebras are known to correspond to continuous fields of simple C*-algebras; in general they are very far from being locally trivial even if one assumes that all their fibres are mutually isomorphic. In the case of continuous trace C^*-algebras, a most satisfactory solution was provided by Dixmier and Douady in [6]; there, all fibres are isomorphic to the compact operators $\mathcal{K}(H)$ on a separable Hilbert space, and the classifying invariant is a certain class in the cohomology group $H^3(X;\mathbb{Z})$.

It seems natural to look for similar results in the case of continuous fields with more complicated fibre algebras. In view of recent progress in Elliott's program to classify nuclear C^*-algebras by K-theory data (cf. [14] and [13]), a most obvious choice of possible fibre algebras would be the so-called strongly self-absorbing C^*-algebras. A crucial property of these algebras is that their unital endomorphism semigroup is weakly homotopy equivalent to their automorphism group. This property is analogous to but still substantially weaker than the property of $\mathcal{K}(H)$ that all its endomorphisms inducing the identity map in K-theory are inner automorphisms.

A unital and separable C^*-algebra $\mathcal{D} \neq \mathbb{C}$ is strongly self-absorbing if there is an isomorphism $\mathcal{D} \stackrel{\cong}{\to} \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the inclusion map $\mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, $d \mapsto d \otimes \mathbf{1}_{\mathcal{D}}$, cf. [12]. Strongly self-absorbing C^*-algebras are known to be simple and nuclear; moreover, they are either purely infinite or stably finite. The only known examples are UHF algebras of infinite type (i.e., every prime number that occurs in the respective supernatural number occurs with infinite multiplicity), the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, the Jiang–Su algebra $\mathcal{Z}$ and tensor products of $\mathcal{O}_\infty$ with UHF algebras of infinite type, see [12]. All these examples are known to be K_1-injective, i.e., the canonical map $U(\mathcal{D})/U_0(\mathcal{D}) \to K_1(\mathcal{D})$ is injective.

We shall state our results in terms of $\mathcal{C}(X)$-algebras; this is a concept generalizing continuous fields of C^*-algebras, cf. [9]. The main result of our paper is the following:

THEOREM 1.1. — *Let A be a separable unital $\mathcal{C}(X)$-algebra over a finite dimensional compact metrizable space X. Suppose that all the fibres of A are isomorphic to the same strongly self-absorbing K_1-injective C^*-algebra $\mathcal{D}$. Then, A and $\mathcal{C}(X) \otimes \mathcal{D}$ are isomorphic as $\mathcal{C}(X)$-algebras.*

In the case where $\mathcal{D} = \mathcal{O}_\infty$ or $\mathcal{D} = \mathcal{O}_2$, the preceding result was already obtained by the first named author in [3]; it is new for UHF algebras of infinite type and for the Jiang–Su algebra. While it would already be quite satisfactory to have this trivialization result for the known strongly self-absorbing examples, it is remarkable that it can be proven without any of the special properties of the *concrete* algebras. In this sense, the theorem further illustrates the importance of strongly self-absorbing algebras for the Elliott program. (In fact, Theorem 1.1 may clearly be regarded as a classification result for $\mathcal{C}(X)$-algebras with strongly self-absorbing fibres, and as such it contributes to the non-simple case of the Elliott conjecture.) This point of view is one of the reasons why we think that not only the theorem, but also the methods developed for its proof are of independent interest. In the subsequent sections we shall therefore present two rather different proofs of Theorem 1.1. The first approach follows the strategy of [3], using the main results of [7] and [5]. The second one follows ideas from Section 4 of [7]. We outline both approaches in Sections 3 and 4.

In [7, Example 4.7], Hirshberg, Rørdam and the second named author constructed examples of $\mathcal{C}(X)$-algebras with $X = \prod_{\mathbb{N}} S^2$ and fibres isomorphic to any prescribed UHF algebra of infinite type, which do not absorb this UHF algebra (and hence cannot be trivial). In [4], the first named author modified this example to construct a separable, unital $\mathcal{C}(X)$-algebra over the Hilbert cube with each fibre isomorphic to $\mathcal{O}_2$, but which does not have trivial K-theory (and hence cannot be trivial either). Therefore, the dimension condition on X in Theorem 1.1 cannot be removed. However, at the present stage, it is not known whether the theorem also fails for infinite dimensional spaces X if $\mathcal{D} = \mathcal{O}_\infty$ or $\mathcal{D} = \mathcal{Z}$.

Theorem 1.1 remains valid if one replaces K_1-injectivity of D by the (apparently weaker) condition that the commutator subgroup $U(\mathcal{D})^c$ of the unitary group of $\mathcal{D}$ is contained in $U_0(\mathcal{D})$, the connected component of 1. Indeed, the only role of K_1-injectivity in the proofs from this paper as well as the proofs of the results from [12], [7] and [5] that we use here is to ensure that approximate unitary equivalence for any two unital $*$-homomorphisms $\mathcal{D} \to \mathcal{D} \otimes A$ is induced by unitaries $(u_n)_n$ in $U_0(\mathcal{D})$. But as noted in [10] and [5] one can always choose the unitaries $(u_n)_n$ in $U(\mathcal{D})^c$. Conversely, it was observed by Kirchberg [10] that if the continuous field consisting of all continuous $f : [0,1] \to \mathcal{D} \otimes \mathcal{D}$ such that $f(0) \in \mathcal{D} \otimes 1$ and $f(1) \in 1 \otimes \mathcal{D}$ is trivial (or just $\mathcal{D}$-stable) then $U(\mathcal{D})^c \subseteq U_0(\mathcal{D})$.

We wish to point out that, in view of Elliott's program, it is not so surprising that a dimension type condition is necessary to obtain a satisfactory classification result. In fact, all the known counterexamples to the Elliott conjecture exhibit high-dimensional behavior, whereas the conjecture has been successfully confirmed for large classes of topologically low-dimensional C^*-algebras, see [11] for an overview.

2. $\mathcal{C}(X)$-Algebras

We recall some facts and notation about $\mathcal{C}(X)$-algebras (introduced in [9]). For our purposes, it will be enough to restrict to compact spaces.

DEFINITION 2.1. — Let A be a C^*-algebra and X a compact Hausdorff space. A is a $\mathcal{C}(X)$-algebra, if there is a unital $*$-homomorphism $\mu\colon \mathcal{C}(X) \to \mathcal{Z}(\mathcal{M}(A))$ from $\mathcal{C}(X)$ to the center of the multiplier algebra of A.

The map μ is called the *structure map*. We will not always write it explicitly. If A is as above and $Y \subset X$ is a closed subset, then

$$J_Y := \mathcal{C}_0(X \setminus Y) \cdot A$$

is a (closed) two-sided ideal of A; we denote the quotient map by π_Y and set

$$A(Y) = A_Y := A/J_Y.$$

A_Y may be regarded as a $\mathcal{C}(X)$-algebra or as a $\mathcal{C}(Y)$-algebra in the obvious way.

If $a \in A$, we sometimes write a_Y for $\pi_Y(a)$. If Y consists of just one point x, we will slightly abuse notation and write A_x (or $A(x)$), J_x, π_x and a_x (or $a(x)$) in place of $A_{\{x\}}$, $J_{\{x\}}$, $\pi_{\{x\}}$ and $a_{\{x\}}$, respectively. We say that A_x is the fibre of A at x. If $\varphi : A \to B$ is a morphism of $\mathcal{C}(X)$-algebras, we denote by φ_Y the corresponding restriction map $A_Y \to B_Y$.

For any $a \in A$ we have

$$\|a\| = \sup\{\|a_x\| \,:\, x \in X\}.$$

Moreover, the function $x \mapsto \|a_x\|$ from X to $\mathbb{R}$ is upper semicontinuous. If the map is continuous for any $a \in A$, then A is said to be a continuous $\mathcal{C}(X)$-algebra. By [3, Lemma 2.3], any unital $\mathcal{C}(X)$-algebra with simple nonzero fibres is automatically continuous. In this case the structure map is injective and hence X is metrizable if A is separable.

3. Proving the main result: The first approach

In this section we give a proof of Theorem 1.1 which follows ideas of [3] and relies on the main absorption result Theorem 4.6 of [7] and on [5, Theorem 2.2]:

THEOREM 3.1. — *Let A and $\mathcal{D}$ be unital C^*-algebras, with $\mathcal{D}$ separable, strongly self-absorbing and K_1-injective. Then, any two unital $*$-homomorphisms $\sigma, \gamma : \mathcal{D} \to A \otimes \mathcal{D}$ are strongly asymptotically unitarily equivalent, i.e. there is a unitary-valued continuous map $u : (0,1] \to \mathcal{U}(A \otimes \mathcal{D})$, $t \mapsto u_t$, with $u_1 = \mathbf{1}_{A\otimes\mathcal{D}}$ and such that $\lim_{t\to 0} \|u_t\sigma(d)u_t^* - \gamma(d)\| = 0$ for all $d \in D$.*