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GENERALISED HERMITE CONSTANTS, VORONOI THEORY AND HEIGHTS ON FLAG VARIETIES

by Bertrand Meyer

Abstract. — This paper explores the study of the general Hermite constant associated with the general linear group and its irreducible representations, as defined by T. Watanabe. To that end, a height, which naturally applies to flag varieties, is built and notions of perfection and eutaxy characterising extremality are introduced. Finally we acquaint some relations (*e.g.*, with Korkine–Zolotareff reduction), upper bounds and computation relative to these constants.

Résumé (*Constantes d'Hermite généralisées, théorie de Voronoï et hauteurs de variétés drapeaux*)

Nous présentons une étude de la constante d'Hermite générale introduite par T. Watanabe et associée au groupe GL_n et ses représentations fortement rationnelles. A cette fin, nous construisons une hauteur qui s'applique naturellement aux variétés drapeaux et nous définissons une notion de perfection et une notion d'eutaxie propres à caractériser l'extrêmalité. Enfin, nous présentons quelques relations (par exemple avec la réduction de Korkine–Zolotareff), majorations et calculs relatifs à ces constantes.

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Introduction

The traditional Hermite constant can be defined by the following formula

$$\gamma_n = \max_A \min_{\substack{x\in\mathbb{Z}^n,\\ x\neq 0}} \frac{A[x]}{(\det(A))^{1/n}} \tag{1}$$

when A runs through the set of all positive definite quadratic forms, or else, from the lattice standpoint, by the equivalent formula

$$\gamma_n = \max_\Lambda \frac{\min \Lambda}{(\det(\Lambda))^{1/n}} \tag{2}$$

where Λ stands for a lattice of $\mathbb{R}^n$.

These constants appear in various areas ; in particular, they account for the highest density one can reach by regularly packing balls of equal radius.

Diverse generalisations of these constants have been set forth, the most accomplished taking the following shape [23]:

$$\gamma_\pi(\|\cdot\|_{\mathbb{A}_k}) = \max_{g\in G(\mathbb{A}_k)^1} \min_{\gamma\in G(k)} \|\pi(g\gamma)x_\pi\|_{\mathbb{A}_k}^{2/[k:\mathbb{Q}]}. \tag{3}$$

In this formula, an algebraic number field k is fixed, as well as a connected reductive algebraic group G. The notation $G(\mathbb{A}_k)^1$ stands for the unimodular part (*i.e.* the intersection of the kernels of the characters of the group $G(\mathbb{A}_k)$). Besides, π is an irreducible strongly rational representation, x_π is a highest weight vector of the representation, $\mathbb{A}_k$ is the ring of the adèles on k and $\|\cdot\|_{\mathbb{A}_k}$ denotes a height on $\mathbb{S}_\pi(k^n)$, the vector space that carries the action of the representation π.

When the representation π is the natural representation of GL_n on k^n (*i.e.* when for any $x \in k^n$ and any $g \in \mathrm{GL}_n(k)$, $\pi(g)x$ is simply $g(x)$), we recover the Hermite–Humbert constant [10] and in particular the traditional Hermite constant expounded above (equation (1)) when in addition k is the rationnal field. Likewise if π is the representation on the exterior power $\bigwedge^d(k^n)$, we get the Rankin-Thunder constant [21], or simply the Rankin constant [17] if in addition k is the field of rationnals.

The constant $\gamma_\pi^{\mathrm{GL}_n}(\|\cdot\|_{\mathbb{A}_k})$ admits also a geometrical interpretation. Indeed, let us define Q_π the (parabolic) subgroup of G which stabilises the line spanned by the highest weight vector x_π. The map

$$g \mapsto \pi(g^{-1})x_\pi \tag{4}$$

provides an embedding of the flag variety $Q_\pi\backslash\mathrm{GL}_n$ into the projective space $\mathbb{P}(\mathbb{S}_\pi(k^n))$. For $A \in \mathrm{GL}_n(\mathbb{A}_k)$, and $\mathscr{D}$ a flag represented by $x \in \mathbb{S}_\pi(k^n)$, one can define the twisted height H_A by $H_A(\mathscr{D}) = \|Ax\|_{\mathbb{A}_k}$. Let us denote by m the sum of the dimensions of the nested spaces of the flag $\mathscr{D}$. Then the generalised

Hermite constant can be read into as the smallest constant C such that for any $A \in \mathrm{GL}_n(\mathbb{A}_k)$, there exists a rational flag $\mathscr{D}$ satisfying

$$H_A(\mathscr{D}) \leqslant C^{1/2} |\det(A)|_{\mathbb{A}_k}^{m/n},$$

which joins up with the definition by J. L. Thunder in [21] as far as subspaces of k^n of fixed dimension are concerned.

In the case of the traditional Hermite constant, G. Voronoï stated two properties, *perfection* and *eutaxy*, which enable to characterise extreme quadratic forms, or in other words, forms that constitute a local maximum of the quotient $\min A/\det(A)^{1/n}$. Generalisations of the notions of eutaxy and perfection have been put forward to fit in the framework of the Rankin [5] or Hermite-Humbert constants [6]. The point of this paper is to define appropriate notions in the case of the constant $\gamma_\pi(\|\cdot\|_{\mathbb{A}_k})$ associated with any irreducible polynomial representation π of the group GL_n.

Our text is organised as follows. In a first part, we fix the conventions we shall stick to in the sequel ; we shall recall what is to be known about irreducible representations of GL_n ; we shall also give a detailed construction of the height that is let invariant by the action of the compact subgroup $K_n(\mathbb{A}_k) = \prod_{v \in \mathfrak{V}_\infty} O_n(k_n) \times \prod_{v \in \mathfrak{V}_f} \mathrm{GL}_n(\mathfrak{o}_v)$ (Think of this subgroup as an adelic analog of the orthogonal group in the real case). In a second part, we shall commit ourselves to exhibit a link between the adelic definition of $\gamma_\pi(\|\cdot\|_{\mathbb{A}_k})$ with an *ad hoc* definition built on Hermite–Humbert forms. This second definition has the advantage of relying only on finitely many places of k: the archimedian places. This allows us, in a third place, to define adequate notions of *perfection* and *eutaxy* for Hermite–Humbert forms and to demonstrate a theorem à la Voronoï. Eventually we bring forth some easy relations, upper bounds and computations relative to the Hermite constants.

1. Representations and heights

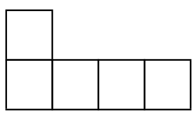
1.1. Conventions. — In the sequel, an integer n is fixed and the algebraic group we shall consider will always be the general linear group $G = \mathrm{GL}_n$.

1.1.1. *Global field.* — The letter k refers to a *number field*, that is an algebraic extension of $\mathbb{Q}$, of degree $d = r_1 + 2r_2$, where r_1 counts its real embeddings and r_2 counts its pairs of complex embeddings. Sometimes, r may designate $r_1 + r_2$. The embedings of k into $\mathbb{R}$ or $\mathbb{C}$ are denoted by $(\sigma_j)_{1 \leqslant j \leqslant r}$, the r_1 first embeddings being real, the r_2 last embeddings being complex. The ring of integers of k will be written $\mathfrak{o}_k$ or simply $\mathfrak{o}$. The field k encompasses h ideal classes, the representative $\mathfrak{a}_1 = \mathfrak{o}$, $\mathfrak{a}_2$, ..., $\mathfrak{a}_h$ of which we fix once for all. The norm of an ideal will be denoted by $\mathcal{N}(\mathfrak{a})$.

1.1.2. *Local fields.* — The set of the places of k is denoted by $\mathfrak{V}$ and divides up into two parts, the set of archimedian or infinite places, denoted by $\mathfrak{V}_\infty$ and the set of ultrametric or finite places, denoted by $\mathfrak{V}_f$. The completion of k (of $\mathfrak{o}$ respectively) at the place v (where v belongs to $\mathfrak{V}$) is denoted by k_v ($\mathfrak{o}_v$ respectively). We shall call d_v the local degree $[k_v : \mathbb{Q}_v]$. The completion k_v is equiped with two absolute values: the absolute value $\|\cdot\|_v$ which is the unique extension of either the absolute value of the real field $\mathbb{Q}_\infty$ when v is an archimedean place or the one of the p-adic field $\mathbb{Q}_p$ when v divides p (that is $\|p\|_v = p^{-1}$), and the normalised absolute value $|\cdot|_v = \|\cdot\|^{d_v}$, which offers the benefit of satisfying the product formula, *i.e.* the equality $\prod_{v\in\mathfrak{V}} |\alpha|_v = 1$ holds for any $\alpha \in k^\times$.

1.1.3. *Partition and related items.* — The letter λ will always refer to a *partition* of any integer m, which we shall note down by $\lambda \vdash m$. Within the borders of this article, we suppose additionnaly that a partition has always less than n parts. Any partition can be depicted by a bar diagram (called Ferrer diagram) drawn in the first quadrant of the plane. The boxes which make up the diagram are indexed by their "cartesian coordinates", the most South–West box being the box $(1,1)$. The symbol $*$ pertains to the transpose partition λ^*, the diagramm of which is by definition the symmetric with respect to the first bissector line of the diagramm of λ. The letters s and t refer to the width and the height of the Ferrer diagramm.

EXAMPLE 1.1. — Let $\lambda = (4,1)$ be the partition $5 = 4 + 1$, its diagramm is

$\lambda =$ 

and the conjugate partition is $\lambda^* = (2,1,1,1)$. Here $s = 4$ and $t = 2$.

With such a partition λ is associated a character χ_λ defined on the torus $(k^\times)^n$ by $\chi_\lambda : (x_1,\dots,x_n) \in (k^\times)^n \mapsto (x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}) \in k^\times$.

When M is a real (respectively complex) vector or square matrix, M' is the transpose (respectively transconjugate) vector or matrix.

1.1.4. *Hermite–Humbert forms.* — We also recall the the space of Hermite–Humbert forms $\mathcal{P}_n(k)$ is by definition the space

$$\mathcal{P}_n(k) = (\mathscr{S}_n^{>0})^{r_1} \times (\mathscr{H}_n^{>0})^{r_2}$$

where $\mathscr{S}_n^{>0}$ denotes the set of determinant 1 symmetric positive definite matrices and $\mathscr{H}_n^{>0}$ the set of determinant 1 Hermitian positive definite matrices.