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QUASI-SEMI-STABLE REPRESENTATIONS

BY XAVIER CARUSO & TONG LIU

ABSTRACT. — Fix K a p-adic field and denote by G_K its absolute Galois group. Let K_∞ be the extension of K obtained by adding p^n-th roots of a fixed uniformizer, and $G_\infty \subset G_K$ its absolute Galois group. In this article, we define a class of p-adic torsion representations of G_∞, called *quasi-semi-stable*. We prove that these representations are "explicitly" described by a certain category of linear algebraic objects. The results of this note should be considered as a first step in the understanding of the structure of quotient of two lattices in a crystalline (resp. semi-stable) Galois representation.

RÉSUMÉ (*Représentations quasi-semi-stables*). — Soient K un corps p-adique et G_K son groupe de Galois absolu. Soit K_∞ l'extension de K obtenue en ajoutant les racines p^n-ièmes d'une uniformisante fixée. Notons $G_\infty \subset G_K$ le groupe de Galois absolu de K_∞. Dans cet article, on définit une classe de représentations p-adiques de torsion du groupe G_∞, que l'on appelle *quasi-semi-stables*. Nous montrons que ces représentations sont « explicitement » décrites *via* une certaine catégories d'objets d'algèbre linéaire. Les résultats dans cette note doivent être considérés comme une première étape dans l'étude de la structure des représentations qui apparaissent comme quotients de deux réseaux d'une représentation galoisienne cristalline (resp. semi-stable).

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Introduction

Let p be a prime number and k a perfect field of characteristic p. Put $W = W(k)$, the ring of Witt vectors with coefficients in k, and $K_0 = \operatorname{Frac} W$. Denote by σ the Frobenius on k, W and K_0. Let K be a totally ramified extension of K_0 of degree e and $\mathcal{O}_K$ its ring of integers. Fix π an uniformizer of $\mathcal{O}_K$. We denote by $\bar{K}$ an algebraic closure of K, by $\mathcal{O}_{\bar{K}}$ its ring of integers and by G_K its absolute Galois group. Fix a sequence (π_n) of elements of $\bar{K}$ satisfying $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. Put $K_n = K(\pi_n)$, $K_\infty = \bigcup_{n\in\mathbb{N}} K_n$ and denote by $G_\infty \subset G_K$ the absolute Galois group of K_∞.

We wish to study representations which can be written as a quotient of two lattices in a crystalline or semi-stable representation. For technical reasons we have to make an assumption on Hodge-Tate weights, namely, they all belong to $\{0,\dots,r\}$ for a nonnegative integer $r < p-1$. The theory of Breuil modules then gives a description of these lattices in term of linear algebra: there exists a category $\mathrm{Mod}_{/S}^{r,\phi,N}$ that is dually equivalent to the category whose objects are these lattices. By mimicking the definition of $\mathrm{Mod}_{/S}^{r,\phi,N}$, one can construct a category of torsion objects $\mathrm{Mod}_{/S_\infty}^{r,\phi,N}$ equipped with a contravariant functor, T_{st}, which takes values in the category of Galois representations. When $er < p-1$, we can prove that $\mathrm{Mod}_{/S_\infty}^{r,\phi,N}$ is an abelian category and T_{st} is fully faithful (see [7]). However, these assertions are false if the assumption $er < p-1$ is removed. In this article, we draw a picture of this structure in a slightly different situation. More precisely, we remove the operator N (that appears in the subscript $\mathrm{Mod}_{/S}^{r,\phi,N}$) and study a new category $\mathrm{Mod}_{/S}^{r,\phi}$. It is endowed with a functor T_{qst} with values in a certain category of G_∞-representations, that we call *quasi-semi-stable*. We define a full subcategory $\mathrm{Max}_{/S_\infty}^{r,\phi}$ and a functor $\mathrm{Max}^r : \mathrm{Mod}_{/S_\infty}^{r,\phi} \to \mathrm{Max}_{/S_\infty}^{r,\phi}$, which is a retraction (and a left adjoint) of the natural inclusion $\mathrm{Max}_{/S_\infty}^{r,\phi} \hookrightarrow \mathrm{Mod}_{/S_\infty}^{r,\phi}$ and which commutes with T_{qst}. We then prove the following (see Theorem 3.7.1 for a more complete statement).

THEOREM 1. — *The category $\mathrm{Max}_{/S_\infty}^{r,\phi}$ is abelian and artinian. Moreover, the restriction of T_{qst} to $\mathrm{Max}_{/S_\infty}^{r,\phi}$ is exact and fully faithful.*

Of course, using duality, we can define the category $\mathrm{Min}_{/S_\infty}^{r,\phi}$ and the functor $\mathrm{Min}^r : \mathrm{Mod}_{/S_\infty}^{r,\phi} \to \mathrm{Max}_{/S_\infty}^{r,\phi}$; they satisfy analogous properties as those stated in Theorem 1. In § 3.6, assuming k to be algebraically closed, we also provide a complete description of simple objects of $\mathrm{Max}_{/S_\infty}^{r,\phi}$, and by duality of $\mathrm{Min}_{/S_\infty}^{r,\phi}$.

If $r = 1$, quasi-semi-stable representations are linked with geometry. In this case, the category $\mathrm{Mod}_{/S_\infty}^{r,\phi}$ is dually equivalent to the category of finite flat group schemes over $\mathcal{O}_K$ killed by a power of p (see [3]). Under this equivalence, the functor Min^r (resp. Max^r) corresponds to the maximal (resp. minimal) models defined by Raynaud in [15]. The following result is then a direct consequence of Theorem 1.

THEOREM 2. — *The category of minimal (resp. maximal) finite flat group schemes over $\mathcal{O}_K$ killed by a power of p is abelian.*

Finally, in the case $r = 1$, we can derive from our results a new proof of the following theorem.

THEOREM 3. — *Let $\mathcal{G}$ and $\mathcal{G}'$ be two finite flat group schemes over $\mathcal{O}_K$ killed by a power of p. Put $T = \mathcal{G}(\bar{K})$, $T' = \mathcal{G}'(\bar{K})$ and consider $f : T \to T'$ a G_∞-equivariant map. Then f is G_K-equivariant.*

Unfortunately, if $r > 1$, quasi-semi-stable representations no longer have a geometric interpretation. Then it is difficult to derive concrete results from Theorem 1 in general. Actually, Theorem 1 should be seen as a preliminary study of the more interesting category $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$; a first part of this work is achieved in [8].

Now, we detail the content of the article. First, we recall definitions of categories of Breuil modules. This allows us to explain more precisely and more clearly our motivations and results. In the second section, we introduce the category $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and we prove that it is equivalent to the category $\mathrm{Mod}^{r,\phi}_{/S_\infty}$. This result is interesting because it will be easier to work with objects of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. Section 3 is devoted to the study of the structure of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} = \mathrm{Mod}^{r,\phi}_{/S_\infty}$: essentially we give a proof of Theorem 1. Then, we assume $r = 1$ and show how the previous results easily imply Theorem 3. The paper ends with some perspectives and open questions.

1. Motivations and settings

In the rest of the paper, we will make an intensive use of Breuil modules, so we gather below all basic definitions about it. The reader may skip it in a first time and come back after when objects are really used.

1.1. Breuil modules. — *Fix a nonnegative integer $r < p-1$.* Recall that π is a fixed uniformizer. Denote by S the p-adic completion of the PD-envelope of $W[u]$ with respect to the kernel of the surjection $W[u] \to \mathcal{O}_K$, $u \mapsto \pi$ (and compatible with the canonical divided powers on $pW[u]$). This ideal is principal generated by $E(u)$, the minimal polynomial of π over K_0. The ring S is endowed with the canonical filtration associated to the PD-envelope and with two endomorphisms:

- a Frobenius ϕ: it is the unique continuous map σ-semi-linear which sends u to u^p
- a monodromy operator N: it is the unique continuous map W-linear that sends u to $-u$ and satisfies $N(xy) = N(x)y + xN(y)$ for all x and y in S (Leibniz rule).

They satisfy $N\phi = p\phi N$. We have $\phi(\mathrm{Fil}^r S) \subset p^r S$ (recall $r < p-1$) and we define $\phi_r = \frac{\phi}{p^r} : \mathrm{Fil}^r S \to S$. Set $c = \phi_1(E(u))$: it is a unit in S.

First, we define a "big" category ${}'\mathrm{Mod}^{r,\phi,N}_{/S}$ whose objects are the following data:

1. a S-module $\mathcal{M}$;
2. a submodule $\mathrm{Fil}^r\mathcal{M} \subset \mathcal{M}$ such that $\mathrm{Fil}^r S\,\mathcal{M} \subset \mathrm{Fil}^r\mathcal{M}$;
3. a ϕ-semi-linear map $\phi_r : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$;
4. a W-linear map $N : \mathcal{M} \to \mathcal{M}$ such that:
   - (Leibniz condition) $N(sx) = sN(x) + N(s)x$ for all $s \in S$, $x \in \mathcal{M}$
   - (Griffiths transversality) $E(u)N(\mathrm{Fil}^r\mathcal{M}) \subset \mathrm{Fil}^r\mathcal{M}$
   - the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Fil}^r\mathcal{M} & \xrightarrow{\ \phi_r\ } & \mathcal{M} \\ {\scriptstyle E(u)N}\downarrow & & \downarrow{\scriptstyle cN} \\ \mathrm{Fil}^r\mathcal{M} & \xrightarrow{\ \phi_r\ } & \mathcal{M} \end{array}$$

Morphisms in ${}'\mathrm{Mod}^{r,\phi,N}_{/S}$ are S-linear maps compatible with Fil^r, ϕ_r and N. There exists in ${}'\mathrm{Mod}^{r,\phi,N}_{/S}$ a notion of exact sequence: a sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is said exact if both sequences $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ and $0 \to \mathrm{Fil}^r\mathcal{M}' \to \mathrm{Fil}^r\mathcal{M} \to \mathrm{Fil}^r\mathcal{M}'' \to 0$ are exact as sequences of S-modules.

Now, we are ready to define full subcategories of ${}'\mathrm{Mod}^{r,\phi,N}_{/S}$. The first one is the category of *strongly divisible modules*, denoted by $\mathrm{Mod}^{r,\phi,N}_{/S}$: it consists of objects $\mathcal{M} \in {}'\mathrm{Mod}^{r,\phi,N}_{/S}$ satisfying the following conditions:

- the module $\mathcal{M}$ is free of finite rank over S;
- the quotient $\mathcal{M}/\mathrm{Fil}^r\mathcal{M}$ has no p-torsion;
- the image of ϕ_r generates $\mathcal{M}$ (as an S-module).