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RADIAL MAXIMAL FUNCTION CHARACTERIZATIONS FOR HARDY SPACES ON RD-SPACES

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Abstract. — An RD-space $\mathcal{X}$ is a space of homogeneous type in the sense of Coifman and Weiss with the additional property that a reverse doubling property holds. The authors prove that for a space of homogeneous type $\mathcal{X}$ having "dimension" n, there exists a $p_0 \in (n/(n+1), 1)$ such that for certain classes of distributions, the $L^p(\mathcal{X})$ quasi-norms of their radial maximal functions and grand maximal functions are equivalent when $p \in (p_0, \infty]$. This result yields a radial maximal function characterization for Hardy spaces on $\mathcal{X}$.

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Résumé (*Caractérisations de fonctions radiales maximales pour les espaces de Hardy sur les RD-espaces*)

Un RD-espace X est un espace de type homogène au sens de Coifman et Weiss, possédant en outre une propriété de doublement inverse. Les auteurs prouvent que pour un espace de type homogène X de « dimension » n, il existe un $p_0 \in (n/(n+1), 1)$ tel que les quasi-normes $L^p(X)$ des fonctions radiales maximales et grand-maximales d'une certaine classe de distributions soient équivalentes lorsque $p \in (p_0, \infty]$. Ce résultat fournit une caractérisation des espaces de Hardy sur X en termes de fonctions radiales maximales.

1. Introduction

The theory of Hardy spaces on Euclidean spaces plays an important role in harmonic analysis and partial differential equations and has been systematically studied and developed; see, for example, [7, 19, 8, 21]. It is well known that spaces of homogeneous type, in the sense of Coifman and Weiss [3], are a natural setting of the Calderón-Zygmund theory of singular integrals; see also [4].

A space of homogeneous type is a set $\mathcal{X}$ equipped with a metric d and a regular Borel measure μ having the doubling property. Coifman and Weiss [4] introduced the atomic Hardy space $H^p_{\rm at}(\mathcal{X})$ for $p \in (0,1]$ and further established a molecular characterization for $H^1_{\rm at}(\mathcal{X})$. Moreover, under the assumption that the measure of any ball in $\mathcal{X}$ is equivalent to its radius (i. e., $\mathcal{X}$ is an Ahlfors 1-regular metric measure space), when $p \in (1/2, 1]$, Macías and Segovia [14] used distributions acting on certain spaces of Lipschitz functions to obtain a grand maximal function characterization for $H^p_{\rm at}(\mathcal{X})$; Han [10] further established a Lusin-area characterization for $H^p_{\rm at}(\mathcal{X})$, and Duong and Yan [6] characterized these atomic Hardy spaces in terms of Lusin-area functions associated with certain Poisson semigroups. Also in this setting, a deep result of Uchiyama [22] states that if $p \in (p_0, 1]$ for some p_0 near 1, for functions in $L^1(\mathcal{X})$, the $L^p(\mathcal{X})$ quasi-norms of the grand maximal functions as in [14] are equivalent to the $L^p(\mathcal{X})$ quasi-norms of the radial maximal functions defined via some kernels in [4].

An important special class of spaces of homogeneous type is called RD-spaces, which is introduced in [12] (see also [11, 15]) and modeled on Euclidean spaces with A_∞-weights (Muckenhoupt's class), Ahlfors n-regular metric measure spaces (see, for example, [13]), Lie groups of polynomial growth (see, for example, [23, 24, 1]) and Carnot-Carathéodory spaces with doubling measure (see, for example, [16, 17, 5, 20, 18]). A Littlewood-Paley theory of Hardy spaces on RD-spaces was established in [11], and these Hardy spaces are shown to coincide with some of Triebel-Lizorkin spaces in [12]. The grand, nontangential and dyadic maximal function characterizations of these Hardy spaces have recently been established in [9].

The main purpose of this paper is twofold: first to generalize the results of Uchiyama [22] to the setting of RD-spaces and second to replace the space $L^1(\mathcal{X})$ used by Uchiyama in [22] by certain spaces of distributions developed in [11, 12]. In other words, we build on the work of Uchiyama [22] to establish a radial maximal function characterization for the Hardy spaces in [11].

To state our main results, we need to recall some definitions and notation. We begin with the classical notions of spaces of homogeneous type ([3], [4]) and RD-spaces ([12]).

Definition 1.1. — *Let $(\mathcal{X}, d)$ be a metric space with a regular Borel measure μ such that all balls defined by d have finite and positive measures. For any $x \in \mathcal{X}$ and $r > 0$, set $B(x,r) \equiv \{y \in \mathcal{X} : d(x,y) < r\}$.*

(i) *The triple $(\mathcal{X}, d, \mu)$ is called a space of homogeneous type if there exists a constant $C_0 \geq 1$ such that for all $x \in \mathcal{X}$ and $r > 0$,*

$$\mu(B(x,2r)) \leq C_0 \mu(B(x,r)) \ (\textit{doubling property}). \tag{1.1}$$

(ii) *Let $0 < \kappa \leq n$. The triple $(\mathcal{X}, d, \mu)$ is called a (κ, n)-space if there exist constants $0 < C_1 \leq 1$ and $C_2 \geq 1$ such that for all $x \in \mathcal{X}$, $0 < r < \operatorname{diam}(\mathcal{X})/2$ and $1 \leq \lambda < \operatorname{diam}(\mathcal{X})/(2r)$,*

$$C_1 \lambda^\kappa \mu(B(x,r)) \leq \mu(B(x,\lambda r)) \leq C_2 \lambda^n \mu(B(x,r)), \tag{1.2}$$

where $\operatorname{diam}(\mathcal{X}) \equiv \sup_{x,\,y \in \mathcal{X}} d(x,y)$.

A space of homogeneous type is called an RD-space, if it is a (κ, n)-space for some $0 < \kappa \leq n$, i. e., some "reverse" doubling condition holds.

Remark 1.2. — (i) *A regular Borel measure μ has the property that open sets are measurable and every set is contained in a Borel set with the same measure; see [13].*

(ii) *The number n in some sense measures the "dimension" of $\mathcal{X}$. Obviously any (κ, n) space is a space of homogeneous type with $C_0 = C_2 2^n$. Conversely, any space of homogeneous type satisfies the second inequality of (1.2) with $C_2 = C_0$ and $n = \log_2 C_0$.*

(iii) *If μ is doubling, then μ satisfies (1.2) if and only if there exist constants $a_0 > 1$ and $\widetilde{C}_0 > 1$ such that for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})/a_0$,*

$$\mu(B(x,a_0 r)) \geq \widetilde{C}_0 \mu(B(x,r)) \quad (\textit{reverse doubling property})$$

(If $a_0 = 2$, this is the classical reverse doubling condition), and equivalently, for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})/a_0$,

$$B(x,a_0 r) \setminus B(x,r) \neq \varnothing;$$

see [12]*. From this, it follows that if* $\mathcal{X}$ *is an RD-space, then* $\mu(\{x\}) = 0$ *for all* $x \in \mathcal{X}$.

Throughout the whole paper, we always assume that $\mathcal{X}$ is an RD-space and $\mu(\mathcal{X}) = \infty$. For any x, $y \in \mathcal{X}$ and $\delta > 0$, set $V_\delta(x) \equiv \mu(B(x,\delta))$ and $V(x,y) \equiv \mu(B(x,d(x,y)))$. It follows from (1.1) that $V(x,y) \sim V(y,x)$. The following notion of approximations of the identity on RD-spaces is a variant of that in [12, Definitions 2.1, 2.2]; see also [11].

DEFINITION 1.3. — *Let* $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$ *and* $\epsilon_3 > 0$*. A sequence* $\{S_k\}_{k\in\mathbb{Z}}$ *of bounded linear integral operators on* $L^2(\mathcal{X})$ *is said to be a special approximation of the identity of order* $(\epsilon_1, \epsilon_2, \epsilon_3)$ *(for short,* $(\epsilon_1, \epsilon_2, \epsilon_3)$*-* SAOTI *), if there exists a constant* $C_3 > \sqrt{2}$ *such that for all* $k \in \mathbb{Z}$ *and all* x, x', y *and* $y' \in \mathcal{X}$, $S_k(x,y)$*, the integral kernel of* S_k *is a function from* $\mathcal{X} \times \mathcal{X}$ *into* $[0,\infty)$ *satisfying*

(i) $S_k(x,y) \le C_3 \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_2}}{(2^{-k}+d(x,y))^{\epsilon_2}}$;

(ii) $|S_k(x,y) - S_k(x',y)| \le C_3 \frac{d(x,x')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}} \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_2}}{(2^{-k}+d(x,y))^{\epsilon_2}}$ *for* $d(x,x') \le (2^{-k} + d(x,y))/2$;

(iii) *Property (ii) holds with* x *and* y *interchanged;*

(iv) $|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \le C_3 \frac{d(x,x')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}} \frac{d(y,y')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}}$ $\times \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_3}}{(2^{-k}+d(x,y))^{\epsilon_3}}$ *for* $d(x,x') \le (2^{-k}+d(x,y))/3$ *and* $d(y,y') \le (2^{-k} + d(x,y))/3$;

(v) $C_3 V_{2^{-k}}(x) S_k(x,x) > 1$ *for all* $x \in \mathcal{X}$ *and* $k \in \mathbb{Z}$;

(vi) $\int_{\mathcal{X}} S_k(x,y)\, d\mu(y) = 1 = \int_{\mathcal{X}} S_k(x,y)\, d\mu(x)$.

We remark that (i) and (v) of Definition 1.3 imply that $C_3 > \sqrt{2}$. The existence of $(\epsilon_1, \epsilon_2, \epsilon_3)$- SAOTI 's was proved in [12, Theorem 2.1].

The following spaces of test functions play a key role in the theory of function spaces on RD-spaces; see [12, 11].

DEFINITION 1.4. — *Let* $x_1 \in \mathcal{X}$, $r \in (0,\infty)$, $\beta \in (0,1]$ *and* $\gamma \in (0,\infty)$*. A function* φ *on* $\mathcal{X}$ *is said to be a test function of type* (x_1, r, β, γ) *if there exists a nonnegative constant* C *such that*

(i) $|\varphi(x)| \le C \frac{1}{V_r(x_1)+V(x_1,x)} \left(\frac{r}{r+d(x_1,x)}\right)^\gamma$ *for all* $x \in \mathcal{X}$;

(ii) $|\varphi(x)-\varphi(y)| \le C \left(\frac{d(x,y)}{r+d(x_1,x)}\right)^\beta \frac{1}{V_r(x_1)+V(x_1,x)} \left(\frac{r}{r+d(x_1,x)}\right)^\gamma$ *for all* x, $y \in \mathcal{X}$ *satisfying* $d(x,y) \le (r + d(x_1,x))/2$.

We denote by $\mathcal{G}(x_1, r, \beta, \gamma)$ *the set of all test functions of type* (x_1, r, β, γ)*. If* $\varphi \in \mathcal{G}(x_1, r, \beta, \gamma)$*, we define its norm by* $\|\varphi\|_{\mathcal{G}(x_1,r,\beta,\gamma)} \equiv \inf\{C : (i)$ *and* (ii) *hold*$\}$*. The space* $\mathcal{G}(x_1, r, \beta, \gamma)$ *is called the space of test functions.*