Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

AGING AND QUENCHED LOCALIZATION FOR RWRE

Nathanaël Enriquez & Christophe Sabot & Olivier Zindy

Tome 137
Fascicule 3

2009

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre national de la recherche scientifique

pages 423-452

Bull. Soc. math. France
137 (3), 2009, p. 423–452

AGING AND QUENCHED LOCALIZATION FOR ONE-DIMENSIONAL RANDOM WALKS IN RANDOM ENVIRONMENT IN THE SUB-BALLISTIC REGIME

by Nathanaël Enriquez, Christophe Sabot & Olivier Zindy

Abstract. — We consider transient one-dimensional random walks in a random environment with zero asymptotic speed. An aging phenomenon involving the generalized Arcsine law is proved using the localization of the walk at the foot of "valleys" of height $\log t$. In the quenched setting, we also sharply estimate the distribution of the walk at time t.

Résumé (*Phénomène de vieillissement et localisation à environnement fixé pour les marches aléatoires en milieu aléatoire uni-dimensionnelles dans le régime sous-balistique*)

Nous considérons les marches aléatoires en milieu aléatoire uni-dimensionnelles, transientes et de vitesse nulle. Un phénomène de vieillissement exprimé en fonction de la loi de l'Arcsinus généralisée est prouvé en utilisant la localisation de la marche au pied de vallées de hauteur $\log t$. Dans le cas où l'environnement est fixé, nous estimons précisément la loi de la position de la marche au temps t.

Texte reçu le 23 juillet 2008, accepté le 5 février 2009

Nathanaël Enriquez, Laboratoire Modal'X, Université Paris 10, 200 Avenue de la République, 92000 Nanterre, France • *E-mail :* nenriquez@u-paris10.fr
Christophe Sabot, Université de Lyon, Université Lyon 1, Institut Camille Jordan, CNRS UMR 5208, 43, Boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France • *E-mail :* sabot@math.univ-lyon1.fr
Olivier Zindy, Laboratoire de Probabilités et Modèles Aléatoires, CNRS UMR 7599, Université Paris 6, 4 place Jussieu, 75252 Paris Cedex 05, France • *E-mail :* olivier.zindy@upmc.fr

2000 Mathematics Subject Classification. — 60K37 ; 60G50, 60J45, 82D30.

Key words and phrases. — Random walks in random environment, aging, quenched localisation.

1. Introduction

One-dimensional random walks in random environment have been the subject of constant interest in physics and mathematics for the last thirty years since they naturally appear in a great variety of situations in physics and biology.

In 1975, Solomon gave, in a seminal work [26], a criterion of transience-recurrence for such walks moving to the nearest neighbors, and shows that three different regimes can be distinguished: the random walk may be recurrent, or transient with a positive asymptotic speed, but it may also be transient with zero asymptotic speed. This last regime, which does not exist among usual random walks, is probably the one which is the less well understood and its study is the purpose of the present paper.

Let us first recall the main existing results concerning the other regimes. In his paper, Solomon computes the asymptotic speed of transient regimes. In 1982, Sinai states, in [25], a limit theorem in the recurrent case. It turns out that the motion in this case is unusually slow. Namely, the position of the walk at time n has to be normalized by $(\log n)^2$ in order to present a non trivial limit. In 1986, the limiting law is characterized independently by Kesten [21] and Golosov [18]. Let us notice here that, beyond the interest of his result, Sinai introduces a very powerful and intuitive tool in the study of one-dimensional random walks in random environment. This tool is the potential, which is a function on $\mathbb{Z}$ canonically associated to the random environment. The potential itself is a usual random walk when the transition probabilities at each site are independent and identically distributed (i.i.d.).

The proof by Sinai of an annealed limit law in the recurrent case is based on a quenched localization result. Namely, a notion of valley of the potential is introduced, as well as an order on the set of valleys. It is then proved that the walk is localized at time t, with a probability converging to 1, around the bottom of the smallest valley of depth bigger than $\log t$ surrounding the origin. An annealed convergence in law of this site normalized by $(\log t)^2$ implies the annealed limiting law for the walk.

In the case of transient random walks in random environment with zero asymptotic speed, the proof of the limiting law by Kesten, Kozlov and Spitzer [22] does not follow this scheme. Therefore an analogous result to Sinai's localization in the quenched setting was missing. As we will see, the answer to this question is more complicated than in the recurrent case but still very explicit.

In the setting of sub-ballistic transient random walks, the valleys we introduce are, like in [13] and [24], related to the excursions of the potential above its past minimum. Now, the key observation is that with a probability converging to 1, the particle at time t is located at the foot of a valley having depth and width of order $\log t$. Therefore, since the walk spends a random time of order t inside a valley of depth $\log t$, it is not surprising that this random walk exhibits an aging phenomenon.

What is usually called aging is a dynamical out-of-equilibrium physical phenomenon observed in disordered systems like spin-glasses at low temperature, defined by the existence of a limit of a given two-time correlation function of the system as both times diverge keeping a fixed ratio between them; the limit should be a non-trivial function of the ratio. It has been extensively studied in the physics literature, see [9] and therein references.

More precisely, in our setting, Theorem 1 expresses that, for each given ratio $h > 1$, the probability that the particle remains confined within the same valley during the time interval $[t, th]$. This probability is expressed in terms of the generalized Arcsine law, which confirms the status of universality ascribed to this law by Ben Arous and Černý in their study of aging phenomena arising in trap models [4].

Recall that the trap model is a model of random walk that was first proposed by Bouchaud and Dean [10, 8] as a toy model for studying this aging phenomenon. In the mathematics literature, much attention has recently been given to the trap model, and many aging result were derived from it, on $\mathbb{Z}$ in [16] and [3], on $\mathbb{Z}^2$ in [7], on $\mathbb{Z}^d$ $(d \geq 3)$ in [5], or on the hypercube in [1, 2]. A comprehensive approach to obtaining aging results for the trap model in various settings was later developed in [6].

Let us finally mention that Theorem 1 generalizes the aging result obtained by heuristic methods of renormalization by Le Doussal, Fisher and Monthus in [23] in the limit case when the bias of the random walk defining the potential tends to 0 (the case when this bias is 0 corresponding to the recurrent regime for the random walk in random environment). The recurrent case, which also leads to aging phenomenon, was treated in the same article and rigorous arguments were later presented by Dembo, Guionnet and Zeitouni in [12].

The second aspect of our work concerns localization properties of the walk and can be considered as the analog of Sinai's localization result in the transient setting. Unlike the recurrent case, the random walk is not localized near the bottom of a single valley. Nevertheless, if one introduces a confidence threshold α, one can say that, asymptotically, at time t, with a probability converging to 1 on the environment, the walk is localized with probability bigger than α around the bottoms of a finite number of valleys having depth of order $\log t$. This number depends on t and on the environment, but is not converging to infinity with t. Moreover, in Theorem 2 and Corollary 1 we sharply estimate the probability for the walk of being at time t in each of these valleys.

2. Notation and main results

Let $\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of i.i.d. random variables taking values in $(0,1)$ defined on Ω, which stands for the random environment. Denote by P the distribution of ω and by E the corresponding expectation. Conditioning on ω (i.e. choosing an environment), we define the random walk in random environment $X = (X_n,\, n \geq 0)$ on $\mathbb{Z}^{\mathbb{N}}$ as a nearest-neighbor random walk on $\mathbb{Z}$ with transition probabilities given by ω: $(X_n,\, n \geq 0)$ is the Markov chain satisfying $X_0 = 0$ and for $n \geq 0$,

$$P_\omega\left(X_{n+1} = x+1 \,|\, X_n = x\right) = \omega_x,$$
$$P_\omega\left(X_{n+1} = x-1 \,|\, X_n = x\right) = 1-\omega_x.$$

We denote by P_ω the law of $(X_n,\, n \geq 0)$ and E_ω the corresponding expectation. We denote by $\mathbb{P}$ the joint law of $(\omega, (X_n)_{n\geq 0})$. We refer to Zeitouni [27] for an overview of results on random walks in random environment. Let us introduce

$$\rho_i := \frac{1-\omega_i}{\omega_i}, \qquad i \in \mathbb{Z}.$$

Our first main result is the following theorem which shows aging phenomenon in the transient sub-ballistic regime.

THEOREM 1. — *Let $\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of independent and identically distributed random variables such that*

(a) *there exists $0 < \kappa < 1$ for which $E\left[\rho_0^\kappa\right] = 1$ and $E\left[\rho_0^\kappa \log^+ \rho_0\right] < \infty$,*
(b) *the distribution of $\log \rho_0$ is non-lattice.*

Then, for all $h > 1$ and all $\eta > 0$, we have

$$\lim_{t\to\infty} \mathbb{P}(|X_{th} - X_t| \leq \eta \log t) = \frac{\sin(\kappa\pi)}{\pi} \int_0^{1/h} y^{\kappa-1}(1-y)^{-\kappa}\, \mathrm{d}y.$$

REMARK 1. — *The statement of Theorem 1 could be improved in the following way: the size of the localization window $\eta \log t$ could be replaced by any positive function $a(t)$ such that $\lim_{t\to\infty} a(t) = +\infty$ and $a(t) = o(t^\kappa)$ (the authors would like to thank Yueyun Hu who raised this question). The extra constraint $a(t) = o(t^\kappa)$ comes from the fact that t^κ is the order of the distance between successive valleys where the RWRE can be localized. We did not write the proof of the theorem in this more general version since it induces several extra technicalities and makes the proof harder to read. Moreover $\eta \log t$ represents an arbitrary portion of a typical valley (which is of size of order $\log t$) where the RWRE can be localized, and is therefore a natural localization window.*