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A BOGOMOLOV PROPERTY FOR CURVES MODULO ALGEBRAIC SUBGROUPS

BY PHILIPP HABEGGER

ABSTRACT. — Generalizing a result of Bombieri, Masser, and Zannier we show that on a curve in the algebraic torus which is not contained in any proper coset only finitely many points are close to an algebraic subgroup of codimension at least 2. The notion of close is defined using the Weil height. We also deduce some cardinality bounds and further finiteness statements.

RÉSUMÉ (*Une propriété de Bogomolov pour des courbes modulo des sous-groupes algébriques*)

En généralisant un résultat de Bombieri, Masser, et Zannier on montre qu'une courbe plongée dans le tore algébrique qui n'est pas contenue dans un translaté d'un sous-groupe algébrique strict n'a qu'un nombre fini de points proches d'un sous-groupe algébrique de codimension au moins 2. La notion de proximité est définie en utilisant la hauteur de Weil. On déduit également des bornes pour la cardinalité et d'autres énoncés de finitude.

1. Introduction

Let X be an irreducible algebraic curve embedded in the algebraic torus $\mathbf{G}_m^n$ and defined over $\overline{\mathbf{Q}}$, an algebraic closure of $\mathbf{Q}$. Bombieri, Masser, and Zannier [5] showed that if X is not contained in the translate of a proper algebraic subgroup, then only finitely many points in X are contained in an algebraic

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subgroup of $\mathbf{G}_m^n$ of dimension $n-2$. The subgroup dimension $n-2$ is best-possible. Their result is related to several general conjectures stated in the mean time by those three authors [7], Pink [19], and Zilber [27].

In this paper we show that only finitely many points in X are close to an algebraic subgroup of dimension $n-2$, where the notion of close is defined with respect to the Weil height. We also give some finiteness results and cardinality bounds for higher dimensional varieties.

All varieties in this paper are defined over $\overline{\mathbf{Q}}$ and will be identified with their set of algebraic points. By irreducible we will always mean geometrically irreducible. For brevity we call the translate of an algebraic subgroup of $\mathbf{G}_m^n$ a *coset* and the translate of an algebraic subgroup of $\mathbf{G}_m^n$ by a torsion point a *torsion coset*. For an integer m with $0 \leq m \leq n$ we define $\mathcal{H}_m$ to be the set of points in $\mathbf{G}_m^n$ that are contained in an algebraic subgroup of dimension at most m; if $m < 0$ we set $\mathcal{H}_m = \varnothing$. With this notation and with X as in the first paragraph, Bombieri, Masser, and Zannier's Theorem states that $X \cap \mathcal{H}_{n-2}$ is finite.

Let $h(\cdot)$ denote the absolute logarithmic Weil height on $\mathbf{G}_m^n$; the precise definition is given in section 2. This height has the important property, usually called Kronecker's Theorem, that it vanishes precisely on the torsion points of $\mathbf{G}_m^n$. For any subset $H \subset \mathbf{G}_m^n$ and any $\epsilon \in \mathbf{R}$ we define the "truncated cone" around H as

$$C(H,\epsilon) = \{ab; a \in H, b \in \mathbf{G}_m^n, h(b) \leq \epsilon(1+h(a))\}.$$

Kronecker's Theorem implies $C(\mathcal{H}_m, 0) = \mathcal{H}_m$.

This definition showed up in the work of Evertse [12]. A special case of his Theorem 5(i) implies that if $X \subset \mathbf{G}_m^n$ is an irreducible curve not equal to a coset and if Γ is the division closure of a finitely generated subgroup of $\mathbf{G}_m^n$, then $X \cap C(\Gamma, \epsilon)$ is finite for $\epsilon > 0$ small enough. Actually Evertse proved a result for X of any dimension. Earlier, Poonen [20] proved a related result in the context of semi-abelian varieties which was then generalized by Rémond [21]. We will study the intersection of subvarieties of $\mathbf{G}_m^n$ with $C(\mathcal{H}_m, \epsilon)$ for small $\epsilon > 0$.

THEOREM 1.1. — *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed algebraic curve defined over $\overline{\mathbf{Q}}$. If X is not contained in a proper coset there exists $\epsilon > 0$ effective and depending only on $h(X)$, $\deg(X)$, and n such that $X \cap C(\mathcal{H}_{n-2}, \epsilon)$ is finite with cardinality bounded effectively in terms of $h(X)$, $\deg(X)$, and n.*

A quite explicit bound for the cardinality is given by (61).

The height $h(X)$ of any irreducible subvariety X of $\mathbf{G}_m^n$ used in this article is the height $h_{\iota|X}$ defined by Philippon on page 346 [18] where ι is the embedding of $\mathbf{G}_m^n$ into projective space $\mathbf{P}^n$ defined in section 2. This height was also used by the author in [15]. The definition of $\deg(X)$, the degree of X, is recalled in section 2.

Theorem 1.1 generalizes Bombieri, Masser, and Zannier's Theorem and also generalizes the Bogomolov property for our curve X. The Bogomolov property (for curves in $\mathbf{G}_m^n$) actually holds more generally and states that if an irreducible curve in $\mathbf{G}_m^n$ is not a torsion coset, then all but finitely many points on this curve have height bounded below by a positive constant. In Theorem 6.2, Zhang [26] proved this and also a higher dimensional analogue. If $n = 2$, Theorem 1.1 actually follows from the Bogomolov property since $C(\mathcal{H}_0, \epsilon)$ is precisely the set of points in $\mathbf{G}_m^n$ with height $\leq \epsilon$. Theorem 1.1 can be viewed as a sort of Bogomolov property for curves modulo subgroups of dimension $n - 2$. We remark that no new proof of the Bogomolov property is given in this article since Theorem 1.1 itself depends on a quantitative version of this property by Amoroso and David [3].

Theorem 1.1 is proved in two steps. First, we apply a Theorem proved by the author [15], see Theorem 7.1 further down, which uniformly bounds the height of points in the intersection $X \cap C(\mathcal{H}_{n-1}, \epsilon)$ if $\epsilon > 0$ is small enough. The second step, done below in Theorem 1.2, consists in showing that a subset of $X \cap C(\mathcal{H}_{n-2}, \epsilon)$ of bounded height is finite if $\epsilon > 0$ is small enough. Theorem 1.1 follows since we already know that $X \cap C(\mathcal{H}_{n-2}, \epsilon) \subset X \cap C(\mathcal{H}_{n-1}, \epsilon)$ has bounded height for small ϵ.

In Theorem 1.2 below we prove a finiteness statement which holds not only for curves but for any irreducible closed subvariety $X \subset \mathbf{G}_m^n$. This is the main technical result of the article, but before we state it we need some definitions.

The set X^{oa} is obtained by removing from X all anomalous subvarieties and X^{ta} is obtained by removing from X all torsion-anomalous subvarieties; see section 2 for the definition of anomalous and torsion-anomalous subvarieties. The sets X^{oa} and X^{ta} were defined by Bombieri, Masser, and Zannier [7] who showed that X^{oa} is Zariski open in X.

For r and n real numbers with $1 \leq r \leq n$, we define

$$\mathfrak{m}(r, n) = n - 2r + 2^{-d}(r(d+2) - n) \text{ with } d = \left[\frac{n-1}{r}\right], \tag{1}$$

here $[x]$ denotes the greatest integer less or equal to x.

Theorem 1.2. — *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety of dimension $r \geq 1$ defined over $\overline{\mathbf{Q}}$. Let $B \geq 1$ and let m be an integer with $m < \mathfrak{m}(r, n)$.*

(i) *If X is not contained in a proper coset there exists $\epsilon > 0$ effective and depending only on B,* $\deg(X)$*, and n such that*

$$\{p \in X \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\}$$

is not Zariski dense in X.

(ii) *For unrestricted X let $\Delta = (B^r \deg(X))^{(n+6)^{4r}2^{nr}}$. There exists $c(n) > 0$ effective and depending only on n such that if $\epsilon \leq (c(n)\Delta)^{-1}$ then*

$$\{p \in X^{\mathrm{oa}} \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\}$$

is finite of cardinality at most $c(n)\Delta$.

A possible choice for ϵ in part (i) is the right-hand side of (48) with s replaced by n. We by no means claim that the hypothesis on ϵ or the cardinality bound in part (ii) are best-possible with respect to any of the involved quantities like B or $\deg(X)$. We do remark that Δ and $c(n)$ are independent of a field of definition or height of X. This uniformity can be used to obtain the following uniform cardinality bound for a simple family of curves:

COROLLARY 1.3. — *Let $\tau \in \overline{\mathbf{Q}}$ and let $X_\tau \subset \mathbf{G}_m^3$ be the curve defined by $(x+1, x+\tau, x-\tau)$ where $x \neq -1, \pm\tau$. There exist $\epsilon > 0$ and an integer N such that $X_\tau \cap C(\mathcal{H}_1, \epsilon)$ is finite with cardinality bounded by N for all $\tau \in \overline{\mathbf{Q}} \backslash \{0, \pm 1\}$.*

Although the corollary could possibly be generalized to more complicated families of curves, our method cannot handle other simple examples such as $(x, x-1, x-\tau)$.

Corollary 1.3 motivates the following two questions. In Theorem 1.1, can ϵ be chosen depending only on $\deg(X)$ and n? In the same theorem, can the cardinality be bounded in function only of $\deg(X)$ and n?

By definition we have $\mathfrak{m}(1, n) = n - 2 + 2^{-(n-1)} > n - 2$ and thus Theorem 1.2 is optimal with respect to the subgroup dimension if X is a curve. But it is likely that the somewhat unnatural function $\mathfrak{m}(r, n)$ does not lead to optimal results if $2 \leq r \leq n-2$. In fact we conjecture that Theorem 1.2(i) holds with $\mathfrak{m}(r, n)$ replaced by $n - r$. If $1 \leq r \leq n$ and if d is as in (1), then $d > \frac{n-1}{r} - 1$, hence $r(d+2) > n - 1 + r$. We conclude $\mathfrak{m}(r, n) > n - 2r$. Therefore one may always take $m = n - 2r$ in Theorem 1.2. Of course this choice is only interesting if $r \leq n/2$. Further down, in Lemma 6.2 we will see that $\mathfrak{m}(r, n) \geq (n-r)/2$ holds for all $1 \leq r \leq n-1$.

Statements related to the ones in Theorem 1.2 were known earlier with $\epsilon = 0$. Work was done in the multiplicative case by Bombieri, Masser, and Zannier (Lemma 8.1 [8]) and in the abelian case by Rémond (Theorem 2.1 [22]). For example by Lemma 8.1 [8] the set of $p \in X^{\mathrm{ta}} \cap \mathcal{H}_{n-r-1}$ with $h(p) \leq B$ is finite. In this result the subgroup dimension $n - r - 1$ is best-possible for any r and finiteness is obtained for X^{ta} instead of the possibly smaller X^{oa}. These earlier finiteness results involved Lehmer-type height lower bounds. In the multiplicative case such a bound gives a positive lower bound for $h(p)$ if $p \in \mathbf{G}_m^n$ is not contained in any proper algebraic subgroup of $\mathbf{G}_m^n$. Typical