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LITTLEWOOD-PALEY DECOMPOSITIONS ON MANIFOLDS WITH ENDS

BY JEAN-MARC BOUCLET

ABSTRACT. — For certain non compact Riemannian manifolds with ends which may or may not satisfy the doubling condition on the volume of geodesic balls, we obtain Littlewood-Paley type estimates on (weighted) L^p spaces, using the usual square function defined by a dyadic partition.

RÉSUMÉ (*Décomposition Littlewood-Paley des variétés à bouts*). — Pour certaines variétés riemanniennes à bouts, satisfaisant ou non la condition de doublement de volume des boules géodésiques, nous obtenons des décompositions de Littlewood-Paley sur des espaces L^p (à poids), en utilisant la fonction carrée usuelle définie *via* une partition dyadique.

1. Introduction

1.1. Motivation and description of the results. — Let $(\mathcal{M}, g)$ be a Riemannian manifold, Δ_g be the Laplacian on functions and dg be the Riemannian measure.

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Consider a dyadic partition of unit, namely choose $\varphi_0 \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(0,+\infty)$ such that

$$1 = \varphi_0(\lambda) + \sum_{k\geq 0} \varphi(2^{-k}\lambda), \quad \lambda \geq 0. \tag{1.1}$$

The existence of such a partition is standard. In this paper, we are basically interested in getting estimates of $||u||_{L^p(\mathscr{M},dg)}$ in terms of $\varphi(-2^{-k}\Delta_g)u$, either through the following square function

$$S_{-\Delta_g}u(\underline{x}) := \left(|\varphi_0(-\Delta_g)u(\underline{x})|^2 + \sum_{k\geq 0} |\varphi(-2^{-k}\Delta_g)u(\underline{x})|^2 \right)^{1/2}, \ \underline{x} \in \mathscr{M}, \tag{1.2}$$

or, at least, through

$$\left(\sum_{k\geq 0} ||\varphi(-2^k\Delta_g)u||^2_{L^p(\mathscr{M},dg)} \right)^{1/2},$$

and a certain remainder term. For the latter, we have typically in mind estimates of the form

$$||u||_{L^p(\mathscr{M},dg)} \lesssim \left(\sum_{k\geq 0} ||\varphi(-2^k\Delta_g)u||^2_{L^p(\mathscr{M},dg)} \right)^{1/2} + ||u||_{L^2(\mathscr{M},dg)}, \tag{1.3}$$

for $p \geq 2$. In the best possible cases, we want to obtain the equivalence of norms

$$||S_{-\Delta_g}u||_{L^p(\mathscr{M},dg)} \approx ||u||_{L^p(\mathscr{M},dg)}, \tag{1.4}$$

which is well known, for $1 < p < \infty$, if $\mathscr{M} = \mathbb{R}^n$ and g is the Euclidean metric (see for instance [12, 13, 15]).

Such inequalities are typically of interest to localize at high frequencies the solutions (and the initial data) of partial differential equations involving the Laplacian such as the Schrödinger equation $i\partial_t u = \Delta_g u$ or the wave equation $\partial_t^2 u = \Delta_g u$, using that $\varphi(-h^2\Delta_g)$ commutes with Δ_g. For instance, estimates of the form (1.3) have been successfully used in [5] to prove Strichartz estimates for the Schrödinger equation on compact manifolds. The article [5] was the first source of inspiration of the present paper, a part which is to prove (1.3) for non compact manifolds. Another motivation came from the fact that, rather surprisingly, we were unable to find in the literature a reference for the equivalence (1.4) in reasonable cases such as asymptotically conical manifolds (the latter is certainly clear to specialists).

We point out that the equivalence (1.4) actually holds on compact manifolds, but (1.3) is sufficient to get Strichartz estimates. Moreover (1.3) is rather robust and still holds in many cases where (1.4) does not. For instance, on asymptotically hyperbolic manifolds where the volume of geodesic balls grows exponentially (with respect to their radii), (1.4) is not expected to hold, but, as a consequence of the results of the present paper, we have nevertheless (1.3). We will briefly recall the application of (1.3) to Strichartz estimates, and more precisely a spatially localized version thereof, after Theorem 1.7.

Littlewood-Paley inequalities on Riemannian manifolds are subjects of intensive studies. There is a vast literature in harmonic analysis studying continuous analogues of the square function (1.2), the so-called Littlewood-Paley-Stein functions defined via integrals involving the Poisson and heat semigroups [13]. An important point is to prove $L^p \to L^p$ bounds related to these square functions (see for instance [8] and [6]). However, as explained above, weaker estimates of the form (1.3) are often highly sufficient for applications to PDEs. Moreover, square functions of the form (1.2) are particularly convenient in microlocal analysis since they involve compactly supported functions of the Laplacian, rather than fast decaying ones. To illustrate heuristically this point, we consider the linear Schrödinger equation $i\partial_t u = \Delta_g u$: if the initial data is spectrally localized at frequency $2^{k/2}$, i.e. $\varphi(-2^{-k}\Delta_g)u(0,.) = u(0,.)$, there is microlocal finite propagation speed stating that the microlocal support (or wavefront set) of $u(t,.)$ is obtained by shifting the one of $u(0,.)$ along the geodesic flow at speed $\approx 2^{k/2}$. This property, which is very useful in the applications, fails if φ is not compactly supported (away from 0). Another similar interest of compactly supported spectral cutoffs for the Schrödinger equation is that high frequency asymptotics like the geometric optics approximation are much easier to obtain for spectrally localized data.

As far as dyadic decompositions associated to non constant coefficients operators are concerned, we have already mentioned [5]. We also have to quote the papers [7] and [10]. In [7], the authors develop a dyadic Littlewood-Paley theory for tensors on compact surfaces with limited regularity (but in low dimension) which is of great interest for nonlinear applications. In [10], the L^p equivalence of norms for dyadic square functions (including small frequencies) associated to Schrödinger operators are proved for a restricted range of p. See also the recent survey [9] for Schrödinger operators on $\mathbb{R}^n$.

In the present paper, we shall use the analysis of $\varphi(-h^2\Delta_g)$ for $h \in (0,1]$, obtained in [1], to derive Littlewood-Paley inequalities on manifolds with ends. We can summarize our results in a model case as follows (see Definition 1.1 for the general manifolds considered here). Assume for simplicity that a neighborhood of infinity of $(\mathscr{M}, g)$ is isometric to $\big((R,\infty)\times S, dr^2 + d\theta^2/w(r)^2\big)$, with $(S, d\theta^2)$ a compact manifold and $w(r) > 0$ a smooth bounded positive function. For instance $w(r) = r^{-1}$ corresponds to conical ends, and $w(r) = e^{-r}$ to hyperbolic ends. We first show that by considering the modified measure $\widetilde{dg} = w(r)^{1-n} dg \approx dr d\theta$ and the associated modified Laplacian

$\widetilde{\Delta}_g = w(r)^{(1-n)/2} \Delta_g w(r)^{(n-1)/2}$, we always have the equivalence of norms

$$||S_{-\widetilde{\Delta}_g} u||_{L^p(\mathcal{M},\widetilde{dg})} \approx ||u||_{L^p(\mathcal{M},\widetilde{dg})},$$

for $1 < p < \infty$, the square function $S_{-\widetilde{\Delta}_g}$ being defined by changing Δ_g into $\widetilde{\Delta}_g$ in (1.2). By giving weighted version of this equivalence, we recover (1.4) when w^{-1} is of polynomial growth. Nevertheless, we emphasize that (1.4) can not hold in general for it implies that $\varphi(-\Delta_g)$ is bounded on $L^p(\mathcal{M}, dg)$ which may fail for instance in the hyperbolic case (see [14]). Secondly, we prove that more robust estimates of the form (1.3) always hold and can be spatially localized (see Theorem 1.7).

Here are the results.

Definition 1.1. — *The manifold $(\mathcal{M}, g)$ is called almost asymptotic if there exist a compact set $\mathcal{K} \Subset \mathcal{M}$, a real number R, a compact manifold S, a function $r \in C^\infty(\mathcal{M}, \mathbb{R})$ and a function $w \in C^\infty(\mathbb{R}, (0, +\infty))$ with the following properties:*

1. *r is a coordinate near $\overline{\mathcal{M} \setminus \mathcal{K}}$ and*

$$r(\underline{x}) \to +\infty, \quad \underline{x} \to \infty,$$

2. *for some $r_{\mathcal{K}} > 0$, there is a diffeomorphism*

$$\mathcal{M} \setminus \mathcal{K} \to (r_{\mathcal{K}}, +\infty) \times S, \tag{1.5}$$

through which the metric reads in local coordinates

$$g = G_{\text{unif}}\left(r, \theta, dr, w(r)^{-1} d\theta\right) \tag{1.6}$$

with

$$G_{\text{unif}}(r, \theta, V) := \sum_{1 \leq j,k \leq n} G_{jk}(r, \theta) V_j V_k, \quad V = (V_1, \dots, V_n) \in \mathbb{R}^n,$$

if $\theta = (\theta_1, \dots, \theta_{n-1})$ are local coordinates on S.

3. *The symmetric matrix $(G_{jk}(r, \theta))_{1 \leq j,k \leq n}$ has smooth coefficients such that, locally uniformly with respect to θ,*

$$\left|\partial_r^j \partial_\theta^\alpha G_{jk}(r, \theta)\right| \lesssim 1, \qquad r > r_{\mathcal{K}}, \tag{1.7}$$

and is uniformly positive definite in the sense that, locally uniformly in θ,

$$G_{\text{unif}}(r, \theta, V) \approx |V|^2, \quad r > r_{\mathcal{K}}, \ V \in \mathbb{R}^n. \tag{1.8}$$

4. *The function w is smooth and satisfies, for all $k \in \mathbb{N}$,*

$$w(r) \lesssim 1, \tag{1.9}$$

$$w(r)/w(r') \approx 1, \quad \text{if } |r - r'| \leq 1 \tag{1.10}$$

$$\left|d^k w(r)/dr^k\right| \lesssim w(r), \tag{1.11}$$