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NONLINEAR SCHRÖDINGER EQUATION ON FOUR-DIMENSIONAL COMPACT MANIFOLDS

BY PATRICK GÉRARD & VITTORIA PIERFELICE

ABSTRACT. — We prove two new results about the Cauchy problem in the energy space for nonlinear Schrödinger equations on four-dimensional compact manifolds. The first one concerns global well-posedness for Hartree-type nonlinearities and includes approximations of cubic NLS on the sphere as a particular case. The second one provides, in the case of zonal data on the sphere, local well-posedness for quadratic nonlinearities as well as a necessary and sufficient condition of global well-posedness for small energy data in the Hamiltonian case. Both results are based on new multilinear Strichartz-type estimates for the Schrödinger group.

RÉSUMÉ (*Équation de Schrödinger non linéaire sur les variétés quadridimensionnelles compactes*)

Nous démontrons deux résultats concernant le problème de Cauchy dans l'espace d'énergie pour des équations de Schrödinger non linéaires sur des variétés compactes de dimension 4. Le premier établit le caractère globalement bien posé pour des seconds membres du type de Hartree et contient comme cas particulier certaines régularisations de l'équation cubique sur la sphère. Le second résultat fournit, dans le cas de données zonales sur la sphère, le caractère localement bien posé pour des seconds membres quadratiques, ainsi qu'une condition nécessaire et suffisante à l'existence globale lorsque les données sont assez petites et que l'équation est hamiltonienne. Chacun de ces résultats est fondé sur de nouvelles estimations multilinéaires du type de Strichartz pour le groupe de Schrödinger.

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1. Introduction

In a recent series of papers ([8], [9], [10], see also [6], [12], [13] for various surveys), Burq-Gérard-Tzvetkov investigated the Cauchy problem for nonlinear Schrödinger equations (NLS) on Riemannian compact manifolds. In [8], Strichartz estimates with fractional loss of derivatives were established for the Schrödinger group. Roughly speaking, these new estimates involve a loss of derivative which is 1/2 more than in the Euclidean case. For instance, for the cubic NLS equation, they led to local well-posedness in H^s for every $s > \frac{d-1}{2}$, where d denotes the space dimension. In contrast, notice that the Euclidean threshold is $\frac{d-2}{2}$ for $d \geq 2$. A second limitation of these estimates is that the loss of derivative is so high that the dual Strichartz inequalities cannot be used efficiently in the Duhamel formulation of the equation, with the bad consequence that, unlike in the Euclidean case, the regularity threshold provided by this analysis does not improve in the case of a subcubic nonlinearity. However, for low dimensions, the estimates of [8] already provided results of global well-posedness, namely for any defocusing polynomial nonlinearity on surfaces, and for cubic defocusing NLS on three-manifolds. Notice that, in the latter case, neither the Lipschitz continuity nor the smoothness of the flow map on the energy space could be established, due to the criticality of the energy threshold $s = 1 = \frac{d-1}{2}$.

These results were improved in [9], [10] for specific manifolds such as spheres, by establishing new multilinear Strichartz inequalities for the Schrödinger group (see also [7]) based on the clustering properties of the spectrum of the Laplacian on such manifolds. In particular, on such manifolds, the regularity threshold for cubic NLS was improved to the Euclidean one $s_c = \frac{d-2}{2}$ if $d \geq 3$ — the case of $\mathbb{S}^2$, for which $s_c = \frac{1}{4}$, being a notable exception. These results generalized to spheres the famous earlier contributions of Bourgain on tori ([4], [3]).

What is the situation on four-dimensional manifolds ? Recall that, on the Euclidean space $\mathbb{R}^4$, the H^1-critical nonlinearity is the cubic one, since $\frac{d-2}{2} = 1$. In [15], [16], [11], the subcubic defocusing equation was solved in $H^1(\mathbb{R}^4)$ using Strichartz estimates, while the case of the cubic one has been settled more recently by Ryckman-Visan in [17]. In contrast, the only available global existence result on a compact four-manifold seems to be the one of Bourgain in [3], which concerns defocusing nonlinearities of the type $|u|u$ and Cauchy data in $H^s(\mathbb{T}^4)$, $s > 1$, and uses strong harmonic analysis specificities of the torus. Let us discuss in slightly more detail the reasons of this difficulty. Firstly, the strategy of [8] only yields local well-posedness in H^s for $s > \frac{d-1}{2} = \frac{3}{2}$, which is far from the H^1 regularity controlled by the energy and the L^2 conservation laws. Secondly, if one appeals to the improved bilinear estimates valid on sphere-like manifolds, local well-posedness in $H^s(\mathbb{S}^4)$ in the cubic or the subcubic case seems to be limited to $s > \frac{d-2}{2} = 1$. In fact, the obstruction to the convergence in the critical space $H^1(\mathbb{S}^4)$ of an iteration scheme for the cubic NLS equation can be made more precise by combining two results from [8] and [9]. Indeed, from Theorem 4 in [8], we know that the estimate

$$\int_0^{2\pi}\int_{\mathbb{S}^4} |e^{it\Delta} f(x)|^4 \, dt \, dx \lesssim \|f\|^4_{H^{1/2}(\mathbb{S}^4)}$$

is wrong, which, by Remark 2.12 in [9], implies that the flow map of cubic NLS cannot be C^3 near the Cauchy data $u_0 = 0$ in $H^1(\mathbb{S}^4)$. Moreover, notice that this phenomenon occurs for zonal data, namely functions depending only on the distance to a fixed point.

The purpose of this paper is to provide further results on four-dimensional manifolds. We shall study two types of NLS equations. In section 2, we study NLS with the following nonlocal nonlinearity,

$$\begin{cases} i\partial_t u + \Delta u = \left((1-\Delta)^{-\alpha}|u|^2\right) u, \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

where $\alpha > 0$. Notice that the homogeneous version of this nonlinearity on the Euclidean space $\mathbb{R}^d$ reads

$$\left(\frac{1}{|x|^{d-2\alpha}} * |u|^2\right) u$$

so that (1) can be seen as a variant of Hartree's equation on a compact manifold. We refer to the book of Cazenave [11] for results on Hartree type equations on $\mathbb{R}^d$. In the case of manifolds, we obtain the following result.

Theorem 1. — *Let (M,g) be a compact Riemannian manifold of dimension 4 and let $\alpha > \frac{1}{2}$. There exists a subspace X of $\mathscr{C}(\mathbb{R}, H^1(M))$ such that, for every $u_0 \in H^1(M)$, the Cauchy problem (1) has a unique global solution $u \in X$. Moreover, in the special case M is the four-dimensional standard sphere $M = \mathbb{S}^4$, the same result holds for all values $\alpha > 0$ of the parameter.*

Remark. — As $\alpha > 0$ tends to 0, the right hand side of (1) tends to $|u|^2 u$, hence (1) can be seen as an approximation of the cubic nonlinear Schrödinger equation. The second part of Theorem 1 states global well-posedness on $H^1(\mathbb{S}^4)$ for such approximations.

In view of Theorem 1, it is natural to ask for which manifolds equation (1) is globally solvable on H^1 for every $\alpha > 0$, as it is the case on $\mathbb{S}^4$. This question is still widely open, and can be answered only in very particular cases. For instance, if all the geodesic curves on M are closed with the same length (see e.g. Besse [2]), it is classical that the clustering properties of the spectrum are the same as on the sphere— see Proposition 4.2 in [8] for more details, and hence the same strategy developed here on $\mathbb{S}^4$ applies. Moreover, it is likely that an adaptation of Bourgain's proof in [4] gives the same result if $M = \mathbb{R}^4/\mathbb{Z}^4$. However, the generalization to a torus $M = \prod_j \mathbb{R}/a_j\mathbb{Z}$ with arbitrary period already looks very delicate (see [1] for similar generalizations in three space dimensions).

The proof of Theorem 1 relies on the combination of conservation laws for equation (1) with the following quadrilinear estimates,

$$\sup_{\tau\in\mathbb{R}}\left|\int_{\mathbb{R}}\int_M \chi(t)\, e^{it\tau}(1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4dxdt\right|$$
$$\leq C(m(N_1,\dots,N_4))^{s_0}\|f_1\|_{L^2(M)}\|f_2\|_{L^2(M)}\|f_3\|_{L^2(M)}\|f_4\|_{L^2(M)},$$

for every $\chi\in\mathscr{C}_0^\infty(\mathbb{R})$, for every $s_0<1$ and for f_1,f_2,f_3,f_4 satisfying

$$\mathbf{1}_{\sqrt{1-\Delta}\in[N_j,2N_j]}(f_j) = f_j,\quad j=1,2,3,4.$$

Here and in the sequel $m(N_1,\dots,N_4)$ denotes the product of the smallest two numbers among N_1,N_2, N_3,N_4. Moreover u_j and f_j are linked by

$$u_j(t,x) = S(t)f_j(x),\ j=1,2,3,4,$$

where $S(t)=e^{it\Delta}$. Notice that, compared to the multilinear estimates used in [10], a frequency variable τ is added to the left hand side of the estimate. The importance of this frequency variable appears in Lemma 1 below, where it is shown that the above quadrilinear estimates imply a quadrilinear estimate of the general expression

$$\int_{\mathbb{R}}\int_M (1-\Delta)^{-\alpha}(u_1\overline{u}_2)u_3\overline{u}_4dxdt$$

in terms of the Bourgain space norms of arbitrary functions u_j's. It would be interesting to know if the smallest value of α for which these estimates (and hence Theorem 1) are valid depends or not on the geometry of M.

In Section 3, we come back to power nonlinearities. Since we want to go below the cubic powers and at the same time we want to use multilinear estimates, we are led to deal with quadratic nonlinearities. In other words, we study the following equations,

$$i\partial_t u + \Delta u = q(u), \tag{2}$$

where $q(u)$ is a homogeneous quadratic polynomial in $u,\overline{u}$

$$q(u) = au^2 + b\overline{u}^2 + c|u|^2.$$

First we introduce the following definition, related to functions on the sphere.

Definition 1. — *Let $d\geq 2$, and let us fix a pole on $\mathbb{S}^d$. We shall say that a function on $\mathbb{S}^d$ is a zonal function if it depends only on the geodesic distance to the pole. We shall denote by $H^s_{\mathrm{zonal}}(\mathbb{S}^d)$ the space of zonal functions in $H^s(\mathbb{S}^d)$.*