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David Rydh

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SUBMERSIONS AND EFFECTIVE DESCENT OF ÉTALE MORPHISMS

BY DAVID RYDH

ABSTRACT. — Using the flatification by blow-up result of Raynaud and Gruson, we obtain new results for submersive and subtrusive morphisms. We show that universally subtrusive morphisms, and in particular universally open morphisms, are morphisms of *effective* descent for the fibered category of étale morphisms. Our results extend and supplement previous treatments on submersive morphisms by Grothendieck, Picavet and Voevodsky. Applications include the universality of geometric quotients and the elimination of noetherian hypotheses in many instances.

RÉSUMÉ (*Submersion et descente effective de morphismes étales*)
On applique le théorème de « platification » de Raynaud et Gruson aux morphismes subtrusifs et obtient le théorème de structure suivant: Tout morphisme universellement subtrusif de présentation finie a un raffinement se factorisant en un recouvrement ouvert suivi d'un morphisme propre. La première application de ce théorème de structure est un théorème de descente *effective*. On montre que tout morphisme universellement subtrusif est un morphisme de descente effective pour la catégorie fibrée des morphismes étales. Ce résultat réduit l'écart entres schémas et espaces algébriques. Par exemple, on peut montrer que des quotients géométriques sont universels dans la catégorie des espaces algébriques. La deuxième application concerne les limites projectives de schémas. On démontre que tout morphisme universellement subtrusif de présentation finie est la limite de morphismes universellement *submersifs* entre schémas noethériens. Il en découle que la classe de morphismes subtrusifs, introduite par Picavet,

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DAVID RYDH, Department of Mathematics, University of California, Berkeley, 970 Evans Hall, Berkeley, CA 94720-3840 USA • *E-mail :* dary@math.berkeley.edu

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est une extension naturelle de la classe de morphismes submersifs entre schémas noethériens. Avec des méthodes semblables on montre aussi un énoncé analogue pour les morphismes universellement ouverts. De plus, on généralise aux espaces algébriques les propriétés fondamentales des topologies h et qfh introduites par Voevodsky.

Introduction

Submersive morphisms, that is, morphisms inducing the quotient topology on the target, appear naturally in many situations such as when studying quotients, homology, descent and the fundamental group of schemes. Somewhat unexpected, they are also closely related to the integral closure of ideals. Questions related to submersive morphisms of *schemes* can often be resolved by topological methods using the description of schemes as locally ringed spaces. Corresponding questions for *algebraic spaces* are significantly harder as an algebraic space is not fully described as a ringed space. The main result of this paper is an effective descent result which bridges this gap between schemes and algebraic spaces.

The first proper treatment of submersive morphisms seems to be due to Grothendieck [19, Exp. IX] with applications to the fundamental group of a scheme. He shows that submersive morphisms are morphisms of descent for the fibered category of étale morphism. He then proves *effectiveness* for the fibered category of quasi-compact and separated étale morphisms in some special cases, e.g., for finite morphisms and universally open morphisms of finite type between noetherian schemes. Our main result consists of several very general effectiveness results extending those of Grothendieck significantly. For example, we show that any universal submersion of noetherian schemes is a morphism of effective descent for quasi-compact étale morphisms. As an application, these effectiveness results imply that strongly geometric quotients are categorical in the category of algebraic spaces [35].

Later on Picavet singled out a subclass of submersive morphisms in [32]. He termed these morphisms *subtrusive* and undertook a careful study of their main properties. The class of subtrusive morphisms is natural in many respects. For example, over a locally noetherian scheme, every submersive morphism is subtrusive. Picavet has also given an example showing that a finitely presented universally submersive morphism is not necessarily subtrusive. In particular, not every finitely presented universally *submersive* morphism is a limit of finitely presented submersive morphisms of noetherian schemes. We will show that every finitely presented universally *subtrusive* morphism is a limit of finitely presented submersive morphisms of noetherian schemes. This is a key result missing in [32] allowing us to eliminate noetherian hypotheses in questions about universal subtrusions of finite presentation. It also shows that the class of subtrusive morphisms is indeed an important and very natural extension of submersive morphisms of noetherian schemes.

A general observation is that in the noetherian setting it is often useful to describe submersive morphisms using the subtrusive property. For example, there is a valuative criterion for submersions of noetherian schemes [25, Prop. 3.7] which rather describes the essence of the subtrusiveness.

Structure theorem. — An important tool in this article is the structure theorems for universally subtrusive morphisms given in §3: Let $f : X \to Y$ be a universally subtrusive morphism of finite presentation. Then there is a morphism $g : X' \to X$ and a factorization of $f \circ g$

$$X' \xrightarrow{f_1} Y' \xrightarrow{f_2} Y$$

where f_1 is fppf and f_2 is proper, surjective and of finite presentation, cf. Theorem 3.10. This is shown using the flatification result of Raynaud and Gruson [34].

We also show that if f is in addition quasi-finite, then there is a similar factorization as above such that f_1 is an open covering and f_2 is finite, surjective and of finite presentation, cf. Theorem 3.11. Combining these results, we show that every universally subtrusive morphism of finite presentation $f : X \to Y$ has a refinement $X' \to Y$ which factors into an open covering f_1 followed by a surjective and proper morphism of finite presentation f_2.

This structure theorem is a generalization to the non-noetherian case of a result of Voevodsky [43, Thm. 3.1.9]. The proof is somewhat technical and the reader without any interest in non-noetherian questions may prefer to read the proof given by Voevodsky which has a more geometric flavor. Nevertheless, our extension is crucial for the elimination of noetherian hypotheses referred to above.

As a first application, we show in Section 4 that universally subtrusive morphisms of finite presentation are morphisms of effective descent for locally closed subsets. This result is not true for universally *submersive* morphisms despite its topological nature.

Effective descent of étale morphisms. — In Section 5 we use the structure theorems of §3 and the proper base change theorem in étale cohomology to prove that

- Quasi-compact universally *subtrusive* morphisms are morphisms of effective descent for quasi-compact étale morphisms. (Theorem 5.17).
- Universally open and surjective morphisms are morphisms of effective descent for étale morphisms. (Theorem 5.19)

In particular, universal submersions between noetherian schemes are morphisms of effective descent for quasi-compact étale morphisms.

Applications. — The effective descent results of §5 have several applications. One is the study of the algebraic fundamental group using morphisms of effective descent for finite étale covers, cf. [19, Exp. IX, §5]. Another application, also the origin of this paper, is in the theory of quotients of schemes by groups. The effective descent results show that strongly geometric quotients are *categorical* in the category of algebraic spaces [35]. This result is obvious in the category of schemes but requires the results of §5 for the extension to algebraic spaces. The third application in mind is similar to the second. Using the effective descent results we can extend some basic results on the h- and qfh-topologies defined by Voevodsky [43] to the category of algebraic spaces. This is done in §§7–8. The h-topology has been used in singular homology [37], motivic homology theories [44] and when studying families of cycles [38]. The h-topology is also related to the integral closure of ideals [9].

Elimination of noetherian hypotheses. — Let S be an inverse limit of affine schemes S_λ. The situation in mind is as follows. Every ring A is the filtered direct limit of its subrings A_λ which are of finite type over $\mathbb{Z}$. The scheme $S = \operatorname{Spec}(A)$ is the inverse limit of the excellent noetherian schemes $S_\lambda = \operatorname{Spec}(A_\lambda)$.

Let $X \to S$ be a finitely presented morphism. Then $X \to S$ descends to a finitely presented morphism $X_\lambda \to S_\lambda$ for sufficiently large λ [17, Thm. 8.8.2]. By this, we mean that $X \to S$ is the base change of $X_\lambda \to S_\lambda$ along $S \to S_\lambda$. If $X \to S$ is proper (resp. flat, étale, smooth, etc.) then so is $X_\lambda \to S_\lambda$ for sufficiently large λ, cf. [17, Thm. 8.10.5, Thm. 11.2.6, Prop. 17.7.8]. Note that the corresponding result for universally open is missing in [17]. As we have mentioned earlier, the analogous result for universally submersive is false.

In Theorem 6.4 we show that if $X \to S$ is *universally subtrusive* then so is $X_\lambda \to S_\lambda$ for sufficiently large λ. We also show the corresponding result for $X \to S$ *universally open*. An easy application of this result is the elimination of noetherian hypotheses in [17, §§14–15]. In particular, every universally open morphism locally of finite presentation has a locally quasi-finite quasi-section, cf. [17, Prop. 14.5.10].

Appendices. — Some auxiliary results are collected in two appendices. In the first appendix we recall the henselian properties of a scheme which is proper over a complete or henselian local ring. These properties follow from the Stein factorization and Grothendieck's existence theorem and constitute a part of the proper base change theorem in étale cohomology. With algebraic spaces