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Francesca Acquistapace & Fabrizio Broglia & José F. Fernando & Jesús M. Ruiz

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ON THE FINITENESS OF PYTHAGORAS NUMBERS OF REAL MEROMORPHIC FUNCTIONS

BY FRANCESCA ACQUISTAPACE, FABRIZIO BROGLIA, JOSÉ F. FERNANDO & JESÚS M. RUIZ

ABSTRACT. — We consider the 17^{th} Hilbert Problem for global real analytic functions in a modified form that involves infinite sums of squares. Then we prove a local-global principle for a real global analytic function to be a sum of squares of global real meromorphic functions. We deduce that an affirmative solution to the 17^{th} Hilbert Problem for global real analytic functions implies the finiteness of the Pythagoras number of the field of global real meromorphic functions, hence that of the field of real meromorphic power series. This measures the difficulty of the 17^{th} Hilbert problem in the analytic case.

RÉSUMÉ (*Sur la finitude des nombres de Pythagore des fonctions méromorphes réelles*)
Nous considérons le 17^{e} problème de Hilbert pour les fonctions analytiques réelles globales sous une forme modifiée faisant intervenir des sommes infinies de carrés. Nous

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FRANCESCA ACQUISTAPACE, Dipartimento di Matematica, Università degli Studi di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy • *E-mail :* acquistf@dm.unipi.it
FABRIZIO BROGLIA, Dipartimento di Matematica, Università degli Studi di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy • *E-mail :* broglia@dm.unipi.it
JOSÉ F. FERNANDO, Departamento de Álgebra, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain • *E-mail :* josefer@mat.ucm.es • *Url :* http://www.mat.ucm.es/~josefer
JESÚS M. RUIZ, Departamento de Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain • *E-mail :* jesusr@mat.ucm.es • *Url :* http://www.mat.ucm.es/~jesusr

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démontrons alors un principe local-global pour qu'une fonction analytique réelle globale soit une somme de carrés de fonctions méromorphes réelles globales. Nous déduisons qu'une solution affirmative au 17e problème de Hilbert pour les fonctions analytiques réelles globales entraîne la finitude du nombre de Pythagore du corps des fonctions méromorphes réelles globales et donc celle du corps des séries méromorphes réelles. Cela donne une mesure de la difficulté du 17e problème de Hilbert dans le cas analytique.

1. Introduction

Of all possible versions of the famous $17^{\rm th}$ Hilbert Problem, that for global real analytic functions is the one that has resisted any substantial progress. As is well known, the problem is whether

Every positive semidefinite global real analytic function $f : \mathbb{R}^m \to \mathbb{R}$ is a sum of squares of global real meromorphic functions.

In this formulation, sums are *finite*. The best result we can state today goes back to the early 1980s: *a positive semidefinite global real analytic function whose zero set is discrete off a compact set is a sum of squares of global real meromorphic functions* ([4] and [14], [11], see also [13] and [2]). Thus the non-compact case remains wide open, except for surfaces ([3], [1], [9]). In this paper we explore some remarkable features that make the non-compact case quite different from the compact one. Recall that the *Pythagoras number* of a ring is the smallest integer p (or $+\infty$) such that any sum of squares in the ring is a sum of p squares. We will prove:

PROPOSITION 1.1. — *Suppose that every positive semidefinite global real analytic function on $\mathbb{R}^m$ is a finite sum of squares of global real meromorphic functions. Then the field of global real meromorphic functions on $\mathbb{R}^m$ has finite Pythagoras number.*

And the same conclusion holds for the field of real meromorphic function germs. Thus, if we can represent every positive semidefinite function as a finite sum of squares (a qualitative matter), we will not encounter sums of arbitrary length (a quantitative matter). This kind of surprise will come out after the consideration of *infinite sums of squares.* Indeed, dealing with analytic functions, convergent infinite sums have a meaning, and they are a more subtle way to produce positive semidefinite functions. In this setting of infinite sums of squares, we localize the obstruction for a function to be a sum of squares at the germ of its zero set. After this sketchy preamble, let us now be more precise.

In what follows, we consider a real analytic manifold $M \subset \mathbb{R}^n$ (which we can suppose embedded as a closed set). This embedding dimension n will appear in various bounds in our results; the dimension of M will be denoted by m.

(1.2) Germs at a non-empty closed set $Z \subset M$**.** — Germs at Z are defined exactly as germs at a point, through neighborhoods of Z in M; we will denote by f_Z the germ at Z of an analytic function f defined in some neighborhood of Z. We have the ring $\mathcal{O}(M_Z)$ of analytic function germs at Z, and its total ring of fractions $\mathcal{M}(M_Z)$, which is the ring of meromorphic function germs at Z. Note that for $Z = M$ we get nothing but global analytic and global meromorphic functions on M, hence definitions and results for germs apply in particular to global functions. If Z is connected, then $\mathcal{O}(M_Z)$ is a domain and $\mathcal{M}(M_Z)$ a field.

As usual, a germ f_Z is *positive semidefinite* when some representative f is positive semidefinite on some neighborhood of Z.

Next, we define infinite sums of squares. The first attempt to use convergent, even uniformly convergent, series of squares cannot work, as, in the real case, uniform convergence does not guarantee analyticity. As we must operate freely with these infinite sums, we must resort to complexification, which on the other hand is customary in real analytic geometry. Thus, we are led to the following:

DEFINITION 1.3. — Let $Z \subset M$ be a non-empty closed set. An *infinite sum of squares of analytic function germs at* Z is a series $\sum_{k\geq 1} f_k^2$, where all $f_k \in \mathcal{O}(M_Z)$, such that:

(i) in some complexification $\widetilde{M}$ of M there is a neighborhood $\mathcal{V}$ of Z on which each f_k extends to a holomorphic function F_k, and
(ii) for every compact set $L \subset \mathcal{V}$, $\sum_{k\geq 1} \sup_L |F_k|^2 < +\infty$.

The condition (ii) is the standard bound one uses to check that a function series is absolutely and uniformly convergent on compact sets. Accordingly, on the neighborhood $V = \mathcal{V} \cap M$ of Z, the infinite sum $\sum_{k\geq 1} f_k^2$ converges and defines a real analytic function f, and hence we have an analytic function germ f_Z; we write $f_Z = \sum_{k\geq 1} f_k^2 \in \mathcal{O}(M_Z)$. Hence, it makes sense to say that an element of the ring $\mathcal{O}(M_Z)$ is a sum of p squares in $\mathcal{O}(M_Z)$, even for $p = +\infty$.

Next, we consider meromorphic functions:

DEFINITION 1.4. — Let $Z \subset M$ be a non-empty closed set. An analytic function germ f_Z is a *sum of* $p \leq +\infty$ *squares of real meromorphic function germs at* Z if there is $g_Z \in \mathcal{O}(M_Z)$ such that $g_Z^2 f_Z$ is a sum of p squares of real analytic function germs at Z. The zero set $\{g_Z = 0\}$ is called the *bad set* of the sum of squares.

The above notion of bad set mimics the terminology introduced in [7], but notice that here it refers to each given sum of squares, not to the function it represents.

We will say that the bad set of a particular representation of a germ f_Z as a sum of squares of real meromorphic function germs is *controlled* if it is contained in the zero set germ $\{f_Z = 0\}$. We will often seek representations with controlled bad set.

There is little need to remark here that the preceding definitions are restrictive in many ways. But this only means that any positive result on the representation of a function as a sum of squares will be stronger than one would have stated naively.

The central result in this paper is the following local-global principle:

THEOREM 1.5. — *Let $f : M \to \mathbb{R}$ be a positive semidefinite global real analytic function, and $Z = \{f = 0\} \neq \varnothing$ its zero set. Suppose that f_Z is a sum of $p \leq +\infty$ squares of real meromorphic function germs. Then, f is a sum of $2^{n-1}p + 1$ squares of global real meromorphic functions with controlled bad set.*

REMARK 1.6. — To check that f_Z is a sum of $p \leq +\infty$ squares it is enough to check that *for every connected component Y of Z the germ f_Y is a sum of p squares.*

Indeed, those connected components Y form a locally finite family of disjoint closed sets of M, hence of any given complexification $\widetilde{M}$ of M. Thus, we can find a locally finite family of disjoint sets $\mathcal{V}_Y \supset Y$, open in $\widetilde{M}$, whose union $\mathcal{V}$ is an open neighborhood of Z in $\widetilde{M}$. The functions on that union $\mathcal{V}$ are defined through their restrictions to the $\mathcal{V}_Y$'s, and convergence on compact sets works fine. Indeed, every compact set $L \subset \mathcal{V}$ is the union of the sets $L \cap \mathcal{V}_Y \neq \varnothing$, which are finite in number and compact; then we have the bound

$$\sup_L |F|^2 \leq \sum_{L\cap\mathcal{V}_Y\neq\varnothing} \sup_{L\cap\mathcal{V}_Y} |F|^2$$

for any function F. □

Our central result, Theorem 1.5, splits into two separate parts. Firstly, what concerns bad sets:

PROPOSITION 1.7. — *Let $Z \subset M$ be a non-empty closed set, and f_Z an analytic function germ which is a sum of $p \leq +\infty$ squares of meromorphic function germs. Then f_Z is a sum of $2^n p$ squares of meromorphic function germs with controlled bad set. The number of squares can be lowered to $2^{n-1}p$ if f_Z vanishes on Z.*