Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 138
Fascicule 3

2010

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 395-413

Bull. Soc. math. France
138 (3), 2010, p. 395–413

WEYL FORMULA WITH OPTIMAL REMAINDER ESTIMATE OF SOME ELASTIC NETWORKS AND APPLICATIONS

by Kaïs Ammari & Mouez Dimassi

Abstract. — We consider a network of vibrating elastic strings and Euler-Bernoulli beams. Using a generalized Poisson formula and some Tauberian theorem, we give a Weyl formula with optimal remainder estimate. As a consequence we prove some observability and stabilization results.

Résumé (*Formule de Weyl avec reste optimal de quelques réseaux élastiques et applications*)

Nous considérons un réseau de cordes et de poutres d'Euler-Bernoulli. En utilisant une formule de Poisson généralisée et un théorème taubérien nous prouvons une formule de Weyl avec reste optimal. Comme conséquence nous prouvons des résultats d'observabilités et de stabilisations.

1. Introduction

In the last years various models of multiple-link flexible structures have been given and developed. The structures which we have in mind consist of finitely many interconnected flexible elements like strings, beams, plates representative

Texte reçu le 20 décembre 2007, révisé le 29 octobre 2008, accepté le 30 octobre 2008

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2000 Mathematics Subject Classification. — 35P20, 93D15, 93D20.

Key words and phrases. — Networks of strings, networks of Euler-Bernoulli beams, Tauberian theorem, Weyl formula.

of trusses, frames, solar panels, antennae deformable mirrors, for more details concerning the models see [12]. "The spectral analysis of such models displays, in addition to its own mathematical interest, control and stabilization problems, see [9, 10, 11, 12], [15, 16, 18] and [1, 3, 4, 5].

First of all, we introduce some notations, which are simply those of [7], [14], we refer to [7] for more details, that is needed to formulate the problem under consideration.

Let Γ be a connected topological graph embedded in $\mathbb{R}^m$, $m \in \mathbb{N}^*$, with n verticies $\mathcal{S} =\{E_i,\, 1 \leq i \leq n\}$ and N edges $\mathcal{A} =\{k_i,\, 1 \leq i \leq N\}$. Each edge k_j is a Jordan curve in $\mathbb{R}^m$ and is assumed to be parametrized by its arc length parameter x_j, such that the parametrizations

$$\pi_j : [0, l_j] \to k_j : x_j \longmapsto \pi_j(x_j)$$

is $C^\nu([0, l_j], \mathbb{R}^m)$ for all $1 \leq j \leq N$.

We now define the C^ν-network G associated with Γ as the union

$$G = \cup_{j=1}^N k_j.$$

The incidence matrix $D = (d_{ij})_{n\times N}$ of Γ is defined by

$$d_{ij} = \begin{cases} 1 \text{ if } \pi_j(l_i) = E_i, \\ -1 \text{ if } \pi_j(0) = E_i, \\ 0 \, \text{otherwise}. \end{cases}$$

The adjacency matrix $\mathcal{E} = (e_{ih})_{n\times n}$ of Γ is given by

$$e_{ih} = \begin{cases} 1 \text{ if there exists an edge } k_{s(i,h)} \ \text{ between } E_i \text{ and } E_h \\ 0 \text{ otherwise.} \end{cases}$$

The valence (*) of the node E_i will be noted $\gamma(E_i)$. There are two types of nodes: the interior nodes $int\, \mathcal{S} =\{E_i \in \mathcal{S};\, \gamma(E_i) > 1\}$ and the boundary nodes $\partial\mathcal{S} =\{E_i \in \mathcal{S} : \gamma(E_i) = 1\}$. In the following we will denote $I_{ext} = \{i \in\{1, \ldots, n\} : \gamma(E_i) = 1\}$ and $I_{int} =\{1, \ldots, n\} \setminus I_{ext}$. We denote by $N_i = \{j \in \{1, \ldots, n\}, E_i \in k_j\}$ the set of edges adjacent to E_i. We remark that if $E_i \in \partial\mathcal{S}$, then N_i is a singleton which is denoted by $\{j_i\}$.

For a function $u : G \to \mathbb{R}$, we set $u_j = u \circ \pi_j : [0, l_j] \to \mathbb{R}$, its restriction to the edge k_j. We further use the abbreviations:

$$u_j(E_i) = u_j(\pi_j^{-1}(E_i)), u_{jx_j^{(n)}}(E_i) = \frac{d^n u_j}{dx_j^n}(\pi_j^{-1}(E_i)),\, n \in \mathbb{N}^*.$$

Finally, differentiations are carried out on each edge k_j with respect to the arc length parameter x_j.

(*) The valence of the node E_i is the cardinal of the set of edges adjacent to E_i.

We consider the following operator Δ_G on the Hilbert space $H = \prod_{j=1}^{N} L^2(0, l_j)$, endowed with the usual product norm.

$$D(\Delta_G) = \{u \in H,\, u_j \in H^2(0, l_j) \text{ satisfying } (1.1) - (1.3)\}$$

$$\Delta_G u = \Big(- u_{jx_j^{(2)}}\Big)_{j=1}^{N}, \ \forall\, u \in D(\Delta_G).$$

If $O = (O_{ih})_{n\times n}$ is the orientation matrix defined by

$$O_{ih} = \begin{cases} 1 \text{ if } k_{s(i,h)} \text{ is directed from } E_i \text{ to } E_h \\ -1 \text{ if } k_{s(i,h)} \text{ is directed from } E_h \text{ to } E_i \\ 0 \text{ else} \end{cases}$$

$$u \text{ is continuous on } G, \tag{1.1}$$

$$\sum_{j=s(i,h)\in N_i} O_{ih}\, u_{jx_j}(E_i) = 0,\ \forall\, i = 1, \dots, n, \tag{1.2}$$

$$u_{j_i}(E_i) = 0,\ \forall\, i \in I_{ext}. \tag{1.3}$$

We study a model of networks of strings and of Euler-Bernoulli beams.
More precisely we consider the following initial problems :
on a finite network, of length L, made of edges k_j, identified to a real interval of length l_j, $j = 1, \dots, N$, (i.e. $L = \sum_{i=1}^{N} l_i$) we consider the eigenvalue problem

$$-\frac{d^2 u_j}{dx_j^2} = \lambda u_j, \quad k_j,\, j = 1, \dots, N, \tag{1.4}$$

$$u \text{ satisfies } (1.1) - (1.3) \tag{1.5}$$

and

$$\frac{d^4 u_j}{dx_j^4} = \lambda u_j, \quad k_j,\, j = 1, \dots, N, \tag{1.6}$$

$$O_{ih} u_{jx_j^{(2)}}(E_i) = O_{ik} u_{lx_l^{(2)}}(E_i), \text{ if } j = s(i,h),\ l = s(i,k), \tag{1.7}$$

$$\sum_{j=s(i,h)\in N_i} O_{ih} u_{jx_j^{(3)}}(E_i) = 0,\ \forall\, i = 1, \dots, n, \tag{1.8}$$

$$u_{j_i}(E_i) = 0,\ u_{j_i, x_{j_i}^{(2)}}(E_i) = 0,\ \forall\, i \in I_{ext}, \tag{1.9}$$

$$u \text{ satisfies } (1.1) - (1.2). \tag{1.10}$$

In the present paper we give some asymptotic Weyl formula of some networks of strings and of Euler-Bernoulli beams.

The plan of the paper is as follows. In the following section we give precise statements of the main results. The two last sections are devoted to some applications and related question.

2. Asymptotics with optimal remainder estimates

Let $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ be the eigenvalues, repeated according to their multiplicity, of the self-adjoint operator Δ_G on a C^2-network G which is defined in Section 1.

We introduce the counting function of eigenvalues

$$N_{\Delta_G}(\lambda) := \#\sigma(\Delta_G) \cap]-\infty, \lambda],$$

where in general $\#A$ denotes the number of elements of A.

Our main result can now be stated as follows.

THEOREM 2.1. — *There exists $\lambda_0 \gg 1$ such that*

$$N_{\Delta_G}(\lambda) = \frac{L}{\pi}\sqrt{\lambda} + \mathcal{O}(1),$$

uniformly on $\lambda \in]\lambda_0, +\infty[$.

Proof. — Let $\chi \in C_0^\infty(] - \epsilon, \epsilon[)$, with $\chi(0) = 1$. We choose $\epsilon > 0$ small enough such that $l_j > \epsilon$ for all $j = 1, \ldots, N$. We may chose χ with the additional property $\widehat{\chi}(t) \geq 0$ and $\widehat{\chi}(0) > 0$. In fact, it suffices to choose $\chi = \psi \star \tilde{\psi}$ for a suitable $\psi \in C_0^\infty$. Here $\tilde{\psi}(t) = \psi(-t)$. We define

$$\mu(\lambda) = \#\{j; \mu_j \leq \lambda\} = \sum_{\mu_j \leq \lambda} (1)$$

where $\mu_j = \sqrt{\lambda_j}$. According to Lemma 4.1 in the Appendix, we have

$$\sum_{j=0}^{\infty} \widehat{\chi}(\lambda + \mu_j) + \sum_{j=0}^{\infty} \widehat{\chi}(\lambda - \mu_j) = 2L\chi(0) + (n - N)\widehat{\chi}(\lambda). \tag{2.1}$$

Since $\widehat{\chi}(0) > 0$, there exists $\delta > 0$ such that $\widehat{\chi}(t) > \frac{1}{2}\widehat{\chi}(0)$ for all $t \in [-\delta, \delta]$. Combining this with the fact that $\widehat{\chi}(t) \geq 0$ for all $t \in \mathbb{R}$, and using (2.1), we obtain

$$\begin{aligned} \#\{j; \mu_j \in [\lambda - \delta, \lambda + \delta]\} &\leq \frac{2}{\widehat{\chi}(0)} \sum_j \Big(\widehat{\chi}(\lambda + \mu_j) + \widehat{\chi}(\lambda - \mu_j)\Big) \\ &\leq \frac{2}{\widehat{\chi}(0)} \Big(2L\chi(0) + (n - N)\widehat{\chi}(\lambda)\Big) = \mathcal{O}(1), \end{aligned} \tag{2.2}$$