Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

EXISTENCE OF GRAPHS WITH SUB EXPONENTIAL TRANSITIONS PROBABILITY DECAY AND APPLICATIONS

Clément Rau

Tome 138
Fascicule 4

2010

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 491-542

Bull. Soc. math. France
138 (4), 2010, p. 491–542

EXISTENCE OF GRAPHS WITH SUB EXPONENTIAL TRANSITIONS PROBABILITY DECAY AND APPLICATIONS

BY CLÉMENT RAU

ABSTRACT. — In this paper, we recall the existence of graphs with bounded valency such that the simple random walk has a return probability at time n at the origin of order $\exp(-n^{\alpha})$, for fixed $\alpha \in [0,1[$ and with Følner function $\exp(n^{\frac{2\alpha}{1-\alpha}})$. This result was proved by Erschler (see [4], [3]); we give a more detailed proof of this construction in the appendix. In the second part, we give an application of the existence of such graphs. We obtain bounds of the correct order for some functional of the local time of a simple random walk on an infinite cluster on the percolation model.

RÉSUMÉ (*Existence de graphes à transitions de probabilités sous-exponentielles et applications*)

Dans cet article, nous rappelons l'existence de graphes à valence finie tels que la probabilité de retour de la marche aléatoire simple soit de l'ordre de $\exp(-n^{\alpha})$, pour $\alpha \in [0,1[$ et tels que la fonction de Følner du graphe soit en $\exp(n^{\frac{2\alpha}{1-\alpha}})$. Ce résultat a été prouvé par Erschler (voir [4], [3]). Une preuve plus détaillée de cette construction est donnée en annexe. Dans une seconde partie, nous donnons une application de l'existence de tels graphes. Nous obtenons des estimées du bon ordre pour certaines fonctionnelles des temps locaux de la marche aléatoire simple sur un amas infini de percolation.

Texte reçu le 7 mai 2007, révisé le 14 février 2008 et le 4 septembre 2008, accepté le 23 septembre 2008

CLÉMENT RAU, Institut Mathématiques de Toulouse, Université Paul Sabatier, route de Narbonne, 31062 Toulouse Cedex 9, France • *E-mail :* `rau@math.ups-tlse.fr` • *Url :* `http://www.math.univ-toulouse.fr/~rau/`

2000 Mathematics Subject Classification. — 60J10, 60K35, 05C90.

Key words and phrases. — Random walk, local time, percolation cluster, isoperimetric inequality, wreath product, generalized wreath product.

1. Introduction and results

A graph G is a couple $(V(G), E(G))$, where $V(G)$ stands for the set of vertices of G and $E(G)$ stands for the set of edges of G. All graphs G which are considered here are infinite and have bounded geometry and we denote by $\nu(g)$ the number of neighbors of g in G.

We study the following random walk X on G defined by:

$$(1)\qquad \begin{cases} X_0 = g, \\ \mathbb{P}(X_{n+1} = b | X_n = a) = \frac{1}{\nu(a)+1}(1_{\{(a,b)\in E(G)\}} + 1_{\{a=b\}}) \end{cases}$$

The random walk X jumps uniformly on the set of points formed by the point where the walker is and his neighbors. Thus X admits reversible measures which are proportional to $m(x) = \nu(x) + 1$.

In this context, the transition probabilities are linked by the isoperimetric profile. For a graph G and for a subset A of G, we introduce the boundary of A relatively to graph G defined by

$$\partial_G A = \{(x, y) \in E(G); x \in A \text{ and } y \in V(G) - A\}.$$

Actually, we will rather work with Følner function to deal with isoperimetry. Let G be a graph, we note Fol_G the Følner function of G defined by:

$$\mathrm{Fol}_G(k) = \min\{|U|; U \subset V(G) \text{ and } \frac{|\partial_G U|}{|U|} \leq \frac{1}{k}\}.$$

If $G' \subset G$ is a subgraph of G, we will use the Følner function of G' relatively to G defined by:

$$\mathrm{Fol}^G_{G'}(k) = \min\{|U|; U \subset V(G) \text{ and } \frac{|\partial_G U|}{|U|} \leq \frac{1}{k}\}.$$

We have the following proposition (see Coulhon [1])

Proposition 1.1. — *Let $m_0 = \inf_{V(U)} m > 0$ and X be the random walk defined by (1). Assume that $\mathrm{Fol}(n) \geq F(n)$ with F a non negative and non decreasing function, then*

$$\sup_{x,y} \mathbb{P}(X_n = y | X_0 = x) \preceq v(n),$$

where v satisfies:

$$\begin{cases} v'(t) = -\frac{v(t)}{8(F^{-1}(4/v(t)))^2}, \\ v(0) = 1/m_0. \end{cases}$$

(We recall that $a_n \preceq b_n$ if there exist constants c_1 and c_2 such that for all $n \geq 0, a_n \leq c_1 b_{c_2 n}$ and $a_n \approx b_n$ if $a_n \preceq b_n$ and $a_n \succeq b_n$.)

For example, we retrieve that in $\mathbb{Z}^d$, the random walk X defined above has transitions decay at time n less than $n^{-d/2}$ and in $\mathcal{F}_2$ the Cayley graph of the free group with two elements, the transition decay of the random walk are less than e^{-n}. A natural question is to know if there exist graphs with intermediate transitions decay. Some others motivations can be found in Section 2.

The answer is given by the following proposition due to Erschler (see [4]).

Proposition 1.2. — *Let $\alpha \in [0;1[$, $F := e^{x^{\frac{2\alpha}{1-\alpha}}}$ and $\sigma(n) := e^{-n^\alpha}$. There exists a graph $D_F = (V(D_F), E(D_F))$ with bounded valency such that:*

(i) $\mathrm{Fol}_{D_F} \approx F$,
(ii) *there exists a point $d_0 \in V(D_F)$ such that, for all $n, p_n^{D_F}(d_0, d_0) \approx \sigma(n)$,*

where $p_n^{D_F}(,)$ stands for the transitions probability of the random walk X defined above when $G = D_F$.

Most important steps of the proof can be found in [4] for $\alpha \geq 1/3$ and in [3] for $\alpha \leq 1/3$. A complete proof is given in the appendix at the end of this paper following arguments of Erschler. Graphs given in the proof, are called wreath products. Note that a recent study of isoperimetry for wreath products on groups has been done by Gromov in [5]. Wreath products would be useful in the next section, so we recall here the definition. Let A a graph and $(B_z)_{z \in V(A)}$ a family of graphs.

Definition 1.3. — *The wreath product of A and $(B_z)_{z\in V(A)}$ is the graph noted by $A \wr (B_z)_{z \in V(A)}$ (or shortly $A \wr B_z$) such that:*

$$V(A \wr B_z) = \{(a, f); a \in A \text{ and } f : A \to \cup_z B_z \text{ with } \mathrm{supp}(f) < \infty$$

and $\forall z \in A, f(z) \in B_z\}$ *and* $E(A \wr B_z) = \{\Big((a,f)(b,g)\Big);\ (f = g$ *and* $(a,b) \in E(A))$ *or* $(a = b$ *and* $\forall x \neq a f(x) = g(x)$ *and* $\big(f(a), g(a)\big) \in E(B_a))\}$.

This graph can be interpreted as follow: imagine there is a lamp in each point a of A such that each point of B_a defined a different intensity of the lamp. The different intensity of each lamp can be represented by a configuration $f : A \to \cup_a B_a$ which encodes the intensity of the lamp at point a by the value $f(a)$. A point in the wreath product is the couple formed by the position of a walker in graph A and the state of each lamp. A particular case is when the graph B_a (called the fiber) is the same for all $a \in A$.

1.1. Example of application of Proposition 1.2. — With the help of graphs D_F given in Proposition 1.2 and with some others good wreath products, we will be able to find upper bound of functional of type: $\mathbb{E}(e^{-\lambda \sum F(L_{x,n},x)})$ where $L_{x,n} = \#\{k \in [0;n]; X_k = x\}$ on the graph $\mathscr{C}^g$ get after a surcritical percolation on edges of $\mathbb{Z}^d$, where edges are kept or removed with respect to Bernoulli independent variables. The points of $\mathscr{C}^g$ are the point of the infinite connected component $\mathscr{C}$ which contains the origin; we will give more details in Section 2. In particular, we will prove the following property:

THEOREM 1.4. — *Consider a simple random walk X on the infinite cluster of $\mathbb{Z}^d$ that contains the origin Q-a.s. on the set $|\mathscr{C}| = +\infty$, and for large enough n we have:*

$$\forall \alpha \in [0,1]\ \mathbb{E}_0^\omega(e^{-\lambda \sum\limits_{z;L_{z;n}>0} L_{z;n}^\alpha} 1_{\{X_n=0\}}) \approx e^{-n^\eta}, \tag{2}$$

$$\forall \alpha > 1/2\ \mathbb{E}_0^\omega(\prod_{z;L_{z;n}>0} L_{z;n}^{-\alpha}\ 1_{\{X_n=0\}}) \approx e^{-n^{\frac{d}{d+2}} ln(n)^{\frac{2}{d+2}}}, \tag{3}$$

where $\eta = \frac{d+\alpha(2-d)}{2+d(1-\alpha)}$.

The constants present in the relation $\approx$ do not depend on the cluster ω.

REMARK 1.5. — *If we take $\alpha = 0$ in Equation (2), we retrieve the Laplace transform of the number of visited points N_n (see [8]),*

$$\mathbb{E}_0^\omega(e^{-\lambda N_n}) \approx e^{-n^{d/d+2}}.$$

In the whole article, C, c are constants which value can evolve from lines to lines.

2. Applications: study of some functionals

2.1. Kind of problems, case of the lattice $\mathbb{Z}^d$. — Recall that for G a graph and X is a simple random walk on G, we note $L_{x,n} = \#\{k \in [0;n]; X_k = x\}$. The question is to estimate functional of type

$$\mathbb{E}_0^\omega(e^{-\lambda \sum\limits_{z;L_{z;n}>0} F(L_{z;n},z)}), \tag{4}$$

where F is a two variables non negative function. The method developped here is due to Erschler and can be applied on general graph G provided the isoperimetric profile on the graph G is known and the function F has some