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Thomas Dedieu

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INTRINSIC PSEUDO-VOLUME FORMS FOR LOGARITHMIC PAIRS

BY THOMAS DEDIEU

ABSTRACT. — We study an adaptation to the logarithmic case of the Kobayashi-Eisenman pseudo-volume form, or rather an adaptation of its variant defined by Claire Voisin, for which she replaces holomorphic maps by holomorphic K-correspondences. We define an intrinsic logarithmic pseudo-volume form $\Phi_{X,D}$ for every pair (X,D) consisting of a complex manifold X and a normal crossing Weil divisor D on X, the positive part of which is reduced. We then prove that $\Phi_{X,D}$ is generically non-degenerate when X is projective and K_X+D is ample. This result is analogous to the classical Kobayashi-Ochiai theorem. We also show the vanishing of $\Phi_{X,D}$ for a large class of log-K-trivial pairs, which is an important step in the direction of the Kobayashi conjecture about infinitesimal measure hyperbolicity in the logarithmic case.

RÉSUMÉ (*Pseudo-formes volumes intrinsèques pour les paires logarithmiques*)
Nous étudions une adaptation au cas logarithmique de la pseudo-forme volume de Kobayashi-Eisenman, ou plutôt une adaptation de sa variante définie par Claire Voisin, pour laquelle elle remplace les applications holomorphes par des K-correspondances holomorphes. Nous définissons une pseudo-forme volume logarithmique intrinsèque $\Phi_{X,D}$ pour toute paire (X,D) constituée d'une variété complexe X et d'un diviseur de Weil à croisements normaux D sur X, dont la partie positive est réduite. Nous prouvons que $\Phi_{X,D}$ est génériquement non dégénérée quand X est projective et K_X+D est ample. Ce résultat est analogue au théorème de Kobayashi-Ochiai classique.

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THOMAS DEDIEU, Université Paul Sabatier, Institut de Mathématiques de Toulouse, 118 route de Narbonne, F-31062 Toulouse Cedex 9, France • *E-mail :* `dedieu@math.jussieu.fr` • *Url :* `http://www.institut.math.jussieu.fr/~dedieu/`

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Nous montrons aussi l'annulation de $\Phi_{X,D}$ pour une grande classe de paires log-K-triviales, ce qui est une étape importante en direction de la conjecture de Kobayashi sur l'hyperbolicité au sens de la mesure infinitésimale dans le cas logarithmique.

Introduction

In the standard non logarithmic case, Kobayashi and Eisenman have defined an intrinsic pseudo-volume form Ψ_X on every complex manifold X ([13]). The definition involves all holomorphic maps from the unit polydisk $\mathbf{D}^n \subset \mathbf{C}^n$ to X. Ψ_X coincides with the Poincaré hyperbolic volume form on X when X is a quotient (by a group acting freely and properly discontinuously) of the unit polydisk $\mathbf{D}^n$. In fact, if X is a smooth curve of genus g, then we have the following dichotomy as a consequence of the Klein-Poincaré uniformization theorem: if $g = 0$ or $g = 1$, then the universal covering of X is $\mathbf{P}^1$ or $\mathbf{C}$, and Ψ_X vanishes; if $g \geqslant 2$, then the universal covering of X is the unit disk $\mathbf{D}$, and Ψ_X is induced by the Poincaré volume form on $\mathbf{D}$. For an n-dimensional manifold X, one expects the situation to follow the same outline. This is in part proved by the Kobayashi-Ochiai theorem ([15]), which states that if X is of general type, then Ψ_X is non degenerate outside a proper closed algebraic subset of X. A variety X such that $\Psi_X > 0$ almost everywhere is said to be infinitesimal measure hyperbolic. On the other hand, Kobayashi conjectured that if X is not of general type, then Ψ_X vanishes on a Zariski open subset of X. The Kobayashi conjecture is proved in the 2-dimensional case for algebraic varieties, using the classification of surfaces (see [9]): Green and Griffiths show that $\Psi_X = 0$ on a dense Zariski open set when X is covered by abelian varieties, and use the fact that algebraic $K3$ surfaces are swept out by elliptic curves.

It is indeed an important step towards the Kobayashi conjecture to show that if X is a Calabi-Yau variety, then Ψ_X vanishes generically. In [24], Claire Voisin defines a new intrinsic pseudo-volume form $\Phi_{X,an}$, which is a variant of Ψ_X, and for which she is able to show that a very wide range of Calabi-Yau varieties satisfy the Kobayashi conjecture (in fact, she shows that the pseudo-volume form $\Phi_{X,an}$ vanishes on these varieties). She also proves a theorem relative to $\Phi_{X,an}$, which is exactly analogous to the Kobayashi-Ochiai theorem. The definition of $\Phi_{X,an}$ is obtained from the definition of Ψ_X by replacing the holomorphic maps from $\mathbf{D}^n$ to X by K-correspondences. A K-correspondence between two complex manifolds X and Y of the same dimension is a closed analytic subset $\Sigma \subset X \times Y$ satisfying the following properties:
(i) the projections $\Sigma \to X$ and $\Sigma \to Y$ are generically of maximal rank on each irreducible component of Σ,

(ii) the first projection $\Sigma \to X$ is proper,
(iii) for every desingularization $\tau : \widetilde{\Sigma} \to \Sigma$, letting $f := \mathrm{pr}_1 \circ \tau$ and $g := \mathrm{pr}_2 \circ \tau$, one has the inequality $R_f \leqslant R_g$ between the ramification divisors of f and g.

(1)
$$\begin{tikzcd} & \widetilde{\Sigma} \arrow[d, "\tau"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\ & \Sigma \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\ X & & Y \end{tikzcd}$$

A K-correspondence Σ has to be seen as the graph of a multivalued map between X and Y. The last condition (iii) ensures the existence of a generalized Jacobian map $(J_{\widetilde{\Sigma}})^T : g^*K_Y \to f^*K_X$. This definition of a K-correspondence, which was introduced in [24], derives from the notion of K-equivalence, for which both projections $\Sigma \to X$ and $\Sigma \to Y$ are birational.

It is nowadays understood that for certain problems, it is more relevant to consider logarithmic pairs (X, D) rather than simply considering varieties. In this situation, one replaces the canonical bundle K_X of X by the log-canonical bundle $K_X(D)$. A first very classical example of this is given by the study of open varieties. If U is a complex manifold, such that there exist a compact variety X and a normal crossing divisor $D \subset X$, such that $U = X \setminus D$, then the study of the pair (X, D) provides a lot of information about U. For example, the Betti cohomology with complex coefficients of U can be computed as the hypercohomology of the logarithmic de Rham complex $\Omega^\bullet_X(\log D)$, see *e.g.* [22]. It has also been made clear, that the minimal model program has to be worked out for pairs, rather than simply for varieties. But the best clue, showing that it is indeed necessary to define an intrinsic pseudo-volume form for logarithmic pairs (analogous to Ψ_X), is perhaps the following.

In [3], Campana shows that to decompose a compact Kähler variety into components of special and hyperbolic types, one necessarily has to consider fibrations with orbifold bases. By definition, a complex manifold X is of special type if there does not exist any non trivial meromorphic fibration $X \dashrightarrow Y$ with orbifold base of general type. Fano and K-trivial manifolds are special, but for every $n > 0$ and $\kappa \in \{-\infty, 0, 1, \ldots, n-1\}$, there exist n-dimensional manifolds X with $\kappa(X) = \kappa$ that are special. If Y is a complex manifold, an orbifold structure on Y is the data of a $\mathbf{Q}$-divisor $\Delta = \sum_j a_j D_j$, where $0 < a_j \leqslant 1$, $a_j \in \mathbf{Q}$, and the D_j are distinct irreducible divisors on Y. The canonical bundle of the orbifold (Y/Δ) is the $\mathbf{Q}$-divisor $K_Y + \Delta$ on Y. If X and Y are smooth complex varieties, and if $f : X \to Y$ is a holomorphic fibration, then the orbifold base of f is $(Y/\Delta(f))$, where

$$\Delta(f) := \sum\nolimits_{D\subset Y}\left(1-\frac{1}{m(f,D)}\right)\cdot D,$$

$m(f,D)$ being the multiplicity of the fiber of f above the generic point of D. Campana constructs for every variety X (or rather for every orbifold (X/D)) a functorial fibration $c_X : X \to C(X)$, the core of X, which is characterized by the fact that the generic fibers are special, and that the orbifold base is hyperbolic. In addition, he conjectures that the Kobayashi pseudo-metric d_X on X (note that this is not the same as the Kobayashi-Eisenman pseudo-volume form) is the pull-back *via* c_X of a pseudo-metric δ_X on the orbifold base of the core.

In this paper, we seek the definition of a pseudo-volume form $\Phi_{X,D}$ on a logarithmic pair (X,D). Let X be a complex manifold of dimension n, and D be a normal crossing Weil divisor on X, the positive part of which is reduced (we say that D is a normal crossing divisor if its support has normal crossings). Note that we do not require that D has a non zero positive part.

THEOREM 1. — (i) *There exists a logarithmic pseudo-volume form $\Phi_{X,D}$ on the pair (X,D),* i.e. *a pseudo-metric on the line bundle $\bigwedge^n T_X(-D)$, satisfying the following functoriality property. Let Y be a complex manifold, and $\nu : Y \to X$ be a proper morphism with ramification divisor R, such that $\nu^*D - R$ is a normal crossing divisor, the positive part of which is reduced. Then we have*

$$\nu^*\Phi_{X,D} = \Phi_{Y,\nu^*D-R} \tag{2}$$

(when ν is not proper, we only get the inequality $\nu^*\Phi_{X,D} \leqslant \Phi_{Y,\nu^*D-R}$).
(ii) *Let D and D' be two normal crossing Weil divisors on X, the respective positive part of which are reduced. If $D \leqslant D'$, then $\Phi_{X,D} \leqslant \Phi_{X,D'}$.*
(iii) *If $D = 0$, then $\Phi_{X,0} = \Phi_{X,an}$.*

This is obtained by following the definition of $\Phi_{X,an}$ in [24]. One replaces the holomorphic maps between $\mathbf{D}^n$ and X in the definition of Ψ_X by log-K-correspondences between $(\mathbf{D}^n, \Delta_k)$ and (X,D), where Δ_k is the divisor given in $\mathbf{D}^n$ by the equation $z_{n-k+1}\cdots z_n = 0$. They are closed analytic subsets $\Sigma \subset \mathbf{D}^n \times X$, satisfying the following three properties : (i) the projections to X and Y are generically of maximal rank on each irreducible component of Σ, (ii) the first projection $\Sigma \to \mathbf{D}^n$ is proper, and (iii) with the same notations as in (1) above ($\tau : \widetilde{\Sigma} \to \Sigma$ is a desingularization, $f = \mathrm{pr}_1 \circ \tau$, and $g = \mathrm{pr}_2 \circ \tau$)

$$R_f - f^*\Delta_k \leqslant R_g - g^*D. \tag{3}$$

The ramification divisor R_f (resp. R_g) is the zero divisor of the section of $K_{\widetilde{\Sigma}} \otimes (f^*K_{\mathbf{D}^n})^{-1}$ (resp. $K_{\widetilde{\Sigma}} \otimes (g^*K_X)^{-1}$) given by the Jacobian map of f (resp.