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GLOBAL EXISTENCE OF SOLUTIONS TO SCHRÖDINGER EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS BELOW H^1

by Sijia Zhong

Abstract. — In this paper, we will study global well-posedness for the cubic defocusing nonlinear Schrödinger equations on the compact Riemannian manifold without boundary, below the energy space, i.e. $s < 1$, under some bilinear Strichartz assumption. We will find some $\tilde{s} < 1$, such that the solution is global for $s > \tilde{s}$.

Résumé (*Existence globale de solutions des équations de Schrödinger sur les variétés riemanniennes compactes en régularité plus faible que* H^1)

Nous nous intéressons dans cet article au caractère bien posé des équations de Schrödinger non-linéaires cubiques défocalisantes sur les variétés riemanniennes compactes sans bord, en régularité H^s, $s < 1$, sous certaines conditions bilinéaires de Strichartz. Nous trouvons un $\tilde{s} < 1$ tel que la solution est globale pour $s > \tilde{s}$.

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1. Introduction

Suppose (M, g) is any compact Riemannian manifold of dimension 2, without boundary. In this paper, we will study the Cauchy problem for the cubic defocusing nonlinear Schrödinger equations posed on M,

$$\begin{cases} iu_t + \triangle u = |u|^2 u \\ u(0,x) = u_0(x) \in H^s(M), \end{cases} \tag{1.1}$$

where the solution u is a complex valued function on $\mathbb{R} \times M$, and $\triangle$ denotes the laplace operator associated to the metric g on M.

There are two conservation laws:

$$L^2\text{-mass} \int_M |u(t,x)|^2 dx = \int_M |u_0(x)|^2 dx, \tag{1.2}$$

and

$$\text{energy } E(u(t)) = \frac{1}{2}\int_M |\nabla u(t)|_g^2 dx + \frac{1}{4}\int_M |u(t,x)|^4 dx = E_0. \tag{1.3}$$

From [10], N. Burq, P. Gérard, and N. Tzvetkov proved that, if bilinear Strichartz estimate $(\mathscr{P}_{s_0})$(see Definition 1.1) is satisfied for some $0 < s_0 < 1$, then the Cauchy problem (1.1) is locally well-posed on $H^s(M)$, $s > s_0$. Thus, as a corollary, if $s \geqslant 1$, combining with the conservation of energy and L^2-mass, the solution is global. The question we are interested in is whether they are global for $s_0 < s < 1$.

First of all, let us see the situation on the whole space $\mathbb{R}^2$. In this case, equation (1.1) is L^2-critical. From [12], the solution is locally well-posed on $H^s(\mathbb{R}^2)$, $s \geqslant 0$, and also by the conservation laws above, it is easy to get the global well-posedness for $s \geqslant 1$. Then, what about $s < 1$?

In 1998, J. Bourgain, by decomposing the initial data into high frequence part and low frequence part, proved that for $\frac{2}{3} < s < 1$, the solution is global. Then in 2002, the I-team (J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao), in [14] introduced the I-method, and improved it to be $\frac{4}{7} < s < 1$. After that, by combining with the refined Morawetz estimate, the result has been improved little by little, and the best result known is $\frac{2}{5} < s < 1$, (see [17], [13], and [15]). Meanwhile, in [15], the authors claimed that by the method available, $\frac{4}{13}$ could be achieved. Recently, in [19], R. Killip, T. Tao and M. Visan proved that, when the initial data is radial, the solution is global for $s \geqslant 0$, which is exactly the optimal one.

Now, let us see what happens on the compact Riemannian manifold without boundary. The first breakthrough was made by J. Bourgain in [5]. He proved that for $\mathbb{T}^2$, it is locally well posed for $s > 0$. Then in [7], he claimed that for $s > \frac{2}{3}$, the solution would be global and this result was proved by D. De Silva, N. Pavlović, G. Staffilani, and N. Tzirakis in [16]. Recently, Akahori in

[1] proved that for the compact manifold without boundary of dimension 2, if $s > \frac{11-7\alpha_1+4\alpha_2}{(1-\alpha_1)+(11-7\alpha_1+4\alpha_2)}$, the solution of (1.1) would be global exists in $H^s(M)$, here (α_1, α_2) is the pair of positive numbers satisfying

$$\sharp\{k \in \mathbb{N} : |\sqrt{\lambda_k} - \mu| \leq A\} \leq C\mu^{\alpha_1} A^{\alpha_2}.$$

For example for all the compact manifold without boundary, the above estimate holds at least with $(\alpha_1, \alpha_2) = (1, 1)$. Particularly, for $\mathbb{T}^2$ and $\mathbb{S}^2$, $(\alpha_1, \alpha_2) = (0, 1)$, and the s corresponding to them is $s > \frac{15}{16}$.

Here, we are also interested in obtaining an abstract result.

Let us give the main condition of this paper, which is the bilinear Strichartz estimates.

Definition 1.1. — *Let $0 \leqslant s_0 < 1$. We say that $S(t) = e^{it\triangle}$, the flow of the linear Schrödinger equation on M stated above, satisfies property $(\mathscr{P}_{s_0})$ if for all dyadic numbers N, L, and u_0, $v_0 \in L^2(M)$ localized on dyadic intervals of order N, L respectively, i.e.*

$$\mathbf{1}_{N\leqslant\sqrt{-\triangle}<2N}(u_0) = u_0, \text{ and } \mathbf{1}_{L\leqslant\sqrt{-\triangle}<2L}(v_0) = v_0, \tag{1.4}$$

the following estimate holds:

$$\|S(t)u_0 S(t)v_0\|_{L^2((0,1)_t\times M)} \leqslant C(\min(N, L))^{s_0}\|u_0\|_{L^2(M)}\|v_0\|_{L^2(M)}. \tag{1.5}$$

In fact, such kind of bilinear Strichartz estimates were established and used by several authors in the context of the wave equations and of the Schrödinger equations. For example, in [10], N. Burq, P. Gérard, N. Tzvetkov showed that for Zoll surface of dimension 2, especially for $\mathbb{S}^2$, $s_0 = \frac{1}{4}+$. Then in [11], they proved that $(\mathscr{P}_{s_0})$ holds for $s_0 = \frac{1}{2}+$ for $\mathbb{S}^3$ and $s_0 = \frac{3}{4}+$ for $\mathbb{S}^2 \times \mathbb{S}^1$. Also the results from J. Bourgain [5], [4], and [8] proved that $s_0 = 0+$ for $\mathbb{T}^2$, $s_0 = \frac{1}{2}+$ for $\mathbb{T}^3$, and $s_0 = \frac{2}{3}+$ for $\tilde{\mathbb{T}}^3 = \mathbb{R}^3/\prod_{j=1}^3(a_j\mathbb{Z})$, where a_j are pairwise irrational numbers. And R. Anton in [2] proved that $s_0 = \frac{3}{4}+$ for general manifolds with boundary and manifolds without boundary equipped with a Lipschitz metric g for dimension $d = 2$, and for dimension 3 for the nontrapping case. Recently, this result is improved by M. D. Blair, H. F. Smith and C. D. Sogge in [3] to be $\frac{2}{3}+$.

Then, the main result of the paper is,

Theorem 1.2. — *Assume that there exists some $s_0 < s < 1$, such that condition $(\mathscr{P}_{s_0})$ holds, then there must be some $s_0 < \tilde{s} < 1$, so that for $s > \tilde{s}$, the solution to the Cauchy problem (1.1) is global, here*

$$\tilde{s} = \frac{\sqrt{2}}{2} + (1 - \frac{\sqrt{2}}{2})s_0. \tag{1.6}$$

Moreover,

$$\|u(T)\|_{H^s} \lesssim T^{\frac{(s-s_0)(1-s)}{2s^2-4s_0s+s_0^2+2s_0-1}+}, \quad \textit{for } T \gg 1. \tag{1.7}$$

REMARK 1.3. — *1. $\tilde{s}$ is monotone increasing with respect to s_0, and for $s_0 \to 1$, $\tilde{s} \to 1$.*
2. For the special case $s_0 = 0+(\mathbb{T}^2)$, $\tilde{s} = \frac{\sqrt{2}}{2} \sim 0.707$. And for $s_0 = \frac{1}{4}+$(Zoll), $\tilde{s} = \frac{2+3\sqrt{2}}{8} \sim 0.78$.
3. When $s \to 1$, $\|u(T)\|_{H^s}$ is controlled by some constant.

Now, we will state the main idea for the proof briefly.

The aim is to imitate the H^1 argument with the energy. Hence we apply some smoothing operator to improve the regularity of the solution u, so that it makes sense for energy. However, the modified energy isn't conserved any more, so the crucial point is to estimate the growth of the energy. But, contrarily from the $\mathbb{R}^2$ case, the Fourier transformation couldn't be extended trivially. Although, we use eigenfunction expansion for M, there are still some obstacles, especially for the case high-high-low-low eigenvalues. Hence, we need to localized the function to some coordinate patch, and use some semiclassical analysis tools to deal with it.

The paper is organized as follows: in Section 2, we will give some notations and lemmas that will be used later, and in Section 3, we will prove the local well posedness for the modified equation. Then in Section 4, 5 and 6, the change of energy would be estimated, the results of which, will help to prove Theorem 1.2 in Section 7. Finally, in Section 8, which is also the appendix, we will prove two important lemmas that appear in the paper.

2. Notations

In this paper, we denote $s+$ for $s+\epsilon$, and $s-$ for $s-\epsilon$, with some constant $\epsilon > 0$ small enough, and by $< \xi >$, $(1+|\xi|^2)^{\frac{1}{2}}$.

$A \lesssim B$ means there is some constant C, such that $A \leqslant CB$, and $A \sim B$ means both $A \lesssim B$ and $B \lesssim A$.

As the spectrum of $\triangle$ is discrete, let $e_k \in L^2(M)$, $k \in \mathbb{N}$, be an orthonormal basis of eigenfunctions of $-\triangle$ associated to eigenvalues μ_k. Denote by P_k the orthogonal projector on e_k. The following space is called Bourgain spaces: