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THE DIXMIER-MOEGLIN EQUIVALENCE AND A GEL'FAND-KIRILLOV PROBLEM FOR POISSON POLYNOMIAL ALGEBRAS

BY K. R. GOODEARL & S. LAUNOIS

ABSTRACT. — The structure of Poisson polynomial algebras of the type obtained as semiclassical limits of quantized coordinate rings is investigated. Sufficient conditions for a rational Poisson action of a torus on such an algebra to leave only finitely many Poisson prime ideals invariant are obtained. Combined with previous work of the first-named author, this establishes the Poisson Dixmier-Moeglin equivalence for large classes of Poisson polynomial rings, including semiclassical limits of quantum matrices, quantum symplectic and euclidean spaces, quantum symmetric and antisymmetric matrices. For a similarly large class of Poisson polynomial rings, it is proved that the quotient field of the algebra (respectively, of any Poisson prime factor ring) is a rational function field $F(x_1, \dots, x_n)$ over the base field (respectively, over an extension field of the base field) with $\{x_i, x_j\} = \lambda_{ij} x_i x_j$ for suitable scalars λ_{ij}, thus establishing a quadratic Poisson version of the Gel'fand-Kirillov problem. Finally, partial solutions to the isomorphism problem for Poisson fields of the type just mentioned are obtained.

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Résumé (*L'équivalence de Dixmier-Moeglin et un analogue du problème de Gel'fand-Kirillov pour certaines algèbres de Poisson polynômiales*)

Nous étudions la structure de certaines algèbres de Poisson polynômiales obtenues comme limites semi-classiques de certaines déformations quantiques d'anneaux de fonctions régulières. Lorsqu'un tore agit rationnellement sur une telle algèbre de Poisson, nous donnons une condition suffisante pour que cette algèbre n'ait qu'un nombre fini d'idéaux premiers de Poisson invariants sous cette action. Ce résultat, combiné à des résultats antérieurs de K.R. Goodearl, permet d'établir l'équivalence de Dixmier-Moeglin pour une large classe d'algèbres de Poisson polynômiales incluant les limites semi-classiques des matrices quantiques, des espaces Euclidiens and symplectiques quantiques, des matrices symétriques et antisymétriques quantiques. De plus, nous démontrons que le corps des fractions de ces algèbres (respectivement, de leurs quotients premiers de Poisson) est un corps de fractions rationnelles $F(x_1, \dots, x_n)$ sur le corps de base (respectivement, sur une certaine extension du corps de base) dont la structure de Poisson est de la forme $\{x_i, x_j\} = \lambda_{ij} x_i x_j$ pour certains scalaires λ_{ij} convenablement choisis. Ce résultat est un analogue quadratique du problème de Gel'fand-Kirillov pour la structure de Poisson de ces corps. Finallement, nous présentons des résultat partiels quant à la classification de tels corps de fractions à isomorphisme (de Poisson) près.

0. Introduction

Many properties of the noncommutative algebras appearing in the world of quantum groups are known, or are conjectured to be, reflected in parallel properties of the Poisson algebras that arise as their semiclassical limits. The present work targets two fundamental properties – the Dixmier-Moeglin equivalence and the quantum Gel'fand-Kirillov conjecture – and establishes Poisson versions of them for large classes of Poisson algebras of the type appearing as semiclassical limits of quantized coordinate rings. More detail follows.

Fix a base field k of characteristic zero throughout. All algebras are assumed to be over k, and all relevant maps (automorphisms, derivations, etc.) are assumed to be k-linear.

0.1. Poisson algebras. — Recall that a *Poisson algebra* (over k) is a commutative k-algebra A equipped with a Lie bracket $\{-,-\}$ which is a derivation (for the associative multiplication) in each variable. We investigate (*iterated*) *Poisson polynomial algebras* over k, that is, polynomial algebras $k[x_1, \dots, x_n]$ equipped with Poisson brackets such that

$$\big\{x_i,\, k[x_1, \dots, x_{i-1}]\big\} \subseteq k[x_1, \dots, x_{i-1}]x_i + k[x_1, \dots, x_{i-1}]$$

for $i = 2, \dots, n$ (see § 1.1 for more detail on the conditions satisfied by such a bracket). Many such Poisson algebras are semiclassical limits of quantum algebras, and these provide our motivation and focus (see Section 2). The

Kirillov-Kostant-Souriau Poisson structure on the symmetric algebra of a finite dimensional Lie algebra $\mathfrak{g}$ can be put in the form of a Poisson polynomial algebra when $\mathfrak{g}$ is completely solvable. This also holds for the basic example of a *Poisson-Weyl algebra*, namely a polynomial algebra $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ equipped with the Poisson bracket such that

$$\{x_i, x_j\} = \{y_i, y_j\} = 0 \qquad \{x_i, y_j\} = \delta_{ij} \tag{1}$$

for all i, j.

0.2. The Dixmier-Moeglin equivalence. — Let A be a noetherian algebra over a field k (here positive characteristic is allowed). As a first approximation to the (usually complicated) representation theory of A, Dixmier raised the question of classifying the primitive ideals of A, that is, the annihilators of simple (left) modules over A. In general, one expects to characterize the primitive ideals of A among its prime ideals either topologically or algebraically as follows. A prime ideal P of A is *locally closed* provided P is a locally closed point of the prime spectrum $\operatorname{spec} A$, and P is *rational* if the center of the Goldie quotient ring $\operatorname{Fract} A/P$ is algebraic over k. One says that the *Dixmier-Moeglin equivalence* holds in A provided the sets of primitive ideals, locally closed prime ideals, and rational prime ideals coincide. This equivalence was first proved by Dixmier [7] and Moeglin [22] for enveloping algebras of finite dimensional complex Lie algebras, and then extended to arbitrary base fields of characteristic zero by Irving and Small [16]. For quantized coordinate rings of semisimple groups, it follows from work of Hodges, Levasseur, Joseph, and Toro [12, 13, 14, 17, 18] (see [10, §2.4] for details). It was established for large classes of other quantized coordinate rings, including the ones of interest in the present work, by Letzter and the first-named author [10] (see also [2, Chapter II.8]).

0.3. The Poisson Dixmier-Moeglin equivalence. — Let A be a Poisson algebra. A *Poisson ideal* of A is any ideal I such that $\{A, I\} \subseteq I$, and a *Poisson prime* ideal is any prime ideal which is also a Poisson ideal. The set of Poisson prime ideals in A forms the *Poisson prime spectrum*, denoted $\operatorname{P.spec} A$, which is given the relative Zariski topology inherited from $\operatorname{spec} A$. Given an arbitrary ideal J of A, there is a largest Poisson ideal contained in J, called the *Poisson core* of J. The *Poisson-primitive* ideals of A are the Poisson cores of the maximal ideals. (One thinks of the Poisson core of an ideal in a Poisson algebra as analogous to the *bound* of a left ideal L in a noncommutative algebra B, that is, the largest two-sided ideal of B contained in L.) The Poisson-primitive ideals in the coordinate ring of a complex affine Poisson variety V are the defining ideals of the Zariski closures of the symplectic leaves in V [3, Lemma 3.5], and they are the key to Brown and Gordon's concept of *symplectic cores* [3, §3.3].

The *Poisson center* of A is the subalgebra

$$Z_p(A) = \{z \in A \mid \{z, -\} \equiv 0\}.$$

For any Poisson prime ideal P of A, there is an induced Poisson bracket on A/P, which extends uniquely to the quotient field $\operatorname{Fract} A/P$ (e.g., [21, Proposition 1.7]). We say that P is *Poisson rational* if the field $Z_p(\operatorname{Fract} A/P)$ is algebraic over k.

By analogy with the Dixmier-Moeglin equivalence, we say that A satisfies the *Poisson Dixmier-Moeglin equivalence* (e.g., [27, pp. 7,8]) provided the following sets coincide:

1. The set of Poisson-primitive ideals in A;
2. The set of locally closed points in $\mathrm{P.spec}\, A$;
3. the set of Poisson rational Poisson prime ideals of A.

If A is an affine (i.e., finitely generated) k-algebra, then (2) $\subseteq$ (1) $\subseteq$ (3) [27, Propositions 1.7, 1.10], so the main difficulty is whether (3) $\subseteq$ (2). No examples are known of affine Poisson algebras for which the Poisson Dixmier-Moeglin equivalence fails.

We obtain this equivalence for Poisson polynomial algebras via the main result of [8], which established it for Poisson algebras with suitable torus actions, as follows.

0.4. Torus actions. — Suppose that H is a group acting on a Poisson algebra A by *Poisson automorphisms* (i.e., k-algebra automorphisms that preserve the Poisson bracket). For each H-stable Poisson prime J of A, set

$$\mathrm{P.spec}_J\, A = \{P \in \mathrm{P.spec}\, A \mid \bigcap_{h \in H} h(P) = J\},$$

the *H-stratum* of $\mathrm{P.spec}\, A$ corresponding to J. These H-strata partition $\mathrm{P.spec}\, A$ as J runs through the H-stable Poisson primes of A.

Now assume that $H = (k^\times)^r$ is an algebraic torus over k. In this case, the action of H on A is called *rational* provided A is generated (as a k-algebra) by H-eigenvectors whose eigenvalues are rational characters of A. (See § 1.2 for the general definition of a rational action of an algebraic group, and [2, Theorem II.2.7] for the equivalence with the above condition in the case of a torus.) In this situation, the Poisson Dixmier-Moeglin equivalence holds when the number of H-stable Poisson prime ideals in A is finite. More precisely, the following result was proved in [8].

THEOREM 0.1. — [8, Theorem 4.3] *Let A be an affine Poisson algebra and $H = (k^\times)^r$ an algebraic torus acting rationally on A by Poisson automorphisms. Assume that A has only finitely many H-stable Poisson prime ideals.*