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João Pedro P. dos Santos

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LIFTING D-MODULES FROM POSITIVE TO ZERO CHARACTERISTIC

by João Pedro P. dos Santos

Abstract. — We study liftings or deformations of D-modules (D is the ring of differential operators from EGA IV) from positive characteristic to characteristic zero using ideas of Matzat and Berthelot's theory of arithmetic D-modules. We pay special attention to the growth of the differential Galois group of the liftings. We also apply formal deformation theory (following Schlessinger and Mazur) to analyze the space of all liftings of a given D-module in positive characteristic. At the end we compare the problems of deforming a D-module with the problem of deforming a representation of a naturally associated group scheme.

Résumé (*Relèvement de D-modules de caractéristique positive en caractéristique nulle*)

Nous étudions des relèvements des D-modules (D est l'anneau des opérateurs différentiels de EGA IV) de la caractéristique positive en caractéristique nulle en utilisant des idées de Matzat et la théorie de descente par Frobenius (pour les D-modules arithmétiques) de Berthelot. Nous prêtons une attention particulière à la croissance du groupe de Galois différentiel du relèvement. Nous appliquons aussi la théorie locale des déformations (d'après Schlessinger et Mazur) pour analyser l'espace local de modules des relèvements. À la fin, nous comparons la théorie des déformations (locales) d'un D-module avec la théorie des déformations (locales) d'une représentation d'un schéma en groupes naturellement associé.

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João Pedro P. dos Santos, Université Paris VI, Institut de Mathématiques de Jussieu, 175, Rue du Chevaleret, 75013 Paris, France • *E-mail :* `dos-santos@math.jussieu.fr` • *Url :* `http://people.math.jussieu.fr/~dos-santos`

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1. Introduction

The present work focuses on deformations of D-modules (stratified modules) from positive characteristic to zero characteristic. Abandoning generality, this can be grasped by the following situation. Let k be an algebraically closed field of characteristic $p>0$, W be its ring of Witt vectors, $\mathscr{O}$ the ring of convergent power series $\sum_i a_i x^i$ with coefficients in W, so that $\mathscr{O}/p\mathscr{O}=k[x]$. We consider a "linear differential system" (or a $D_{k[x]/k}$-module structure on $k[x]^{\oplus\mu}$)

$$\partial_q y_i = \sum_{j=1}^{\mu} \overline{a}(i,j,q) y_j \tag{1}$$

where $\overline{a}(i,j,q)\in\mathscr{O}/p\mathscr{O}$, ∂_q is the differential operator of order q analogous to $\frac{1}{q!}\frac{\partial^q}{\partial x^q}$ and the matrices $(\overline{a}(i,j,q))_{i,j}$ are required to satisfy certain compatibilities arising from the relations between the various ∂_q. Then we can ask if there is a lifting of these matrices to $\mathscr{O}$ giving rise to a linear differential system. Furthermore, it is reasonable to require that the differential Galois group (DGG) of the lifted system bears resemblance to the differential Galois group of (1). It is the latter question that the present work sets out to analyze (in greater generality). The analysis runs in two distinct directions corresponding to a natural division of the text into two main parts. The first one, comprising sections 3 to 5 deals with the problem of finding such a lifting (with the property concerning the DGG mentioned before). The second part, which occupies sections 6 to 9, deals with the quantitative nature of these liftings or, more precisely, studies the associated deformation problem as understood and proposed as a theory by Schlessinger [38] (and named "a scientific approach" by Kontsevich). Of course, the idea to threat the problem like this comes from Mazur [28]. We now briefly summarize the contents of each section.

In section 2 we review some standard material concerning monoidal categories and torsors. The categories shaping this article are monoidal categories of D-modules and the algebraic Geometry (commutative Algebra, rather) in a monoidal category plays an important conceptual rôle: we talk about algebras, groups, torsors, comodules etc. These and some minor folkloric results will be discussed in section 2 in order to be applied further on.

The existence of section 3 is justified by its expository nature – we fix relevant notations concerning the Frobenius morphism – and by the explicit constructions made in 3.2.2. The main result, Theorem 11, is not proved or commented on and the work is left to [6]; section 3.2.2 will give an operational view of the theory. The principal cognitive gain the reader should look for in section 3 is the understanding that, like D-modules in positive characteristic (after [13, Thm. 1.3]), D-modules in mixed characteristic can be controlled by certain "Frobenius divisions" (see Definition 10). This important observation, in the present context, is due to Matzat and van der Put [27], [26]; in a more general context it is an application of Berthelot's robust theory of Frobenius descent and arithmetic D-modules [3], [4] and [5].

In section 4 we use the concepts from commutative algebra in a monoidal category (§2.3) to develop a "torsor" version of the ideas in section 3. Since the work of Nori [31], it is clear that to a theory of objects in a monoidal category, there should exist a theory of torsors (see also [36]). The application of this principle to the case of D-modules (see Definition 14) is important to the handling of the problem concerning liftings with controlled DGGs as we propose first to find *liftings of the torsors* (see Theorem 16) and then, by lifting the representation, obtain the desired D-modules in characteristic zero by twisting (§2.4.3). The existence of liftings of the torsors (see Proposition 13, Corollary 15 and Theorem 16) comes fairly mechanically from the amount of *elbow room* left by Grothendieck once we incorporate the ideas of Section 3.

Section 5 collects the fruits of the previous ones. Its main result is the existence of liftings of D-modules from positive to zero characteristic whose differential Galois group is "close" to the one in positive characteristic, see Theorem 17. Here the term "close" should be interpreted in the following way. A lifting $\mathscr{V}$ of a D-module in positive characteristic $\mathscr{V}_0$ has an integral differential Galois group (see [37] and its references) which contains the DGG of $\mathscr{V}_0$ on its closed fibre. Then "close" should mean that the DGG of $\mathscr{V}_0$ *is* the closed fibre. To repeat what was said above, the construction of the liftings of the torsor allows us to pass from the problem of finding a lifting of a D-module to the problem of deforming a representation. The principal result, Theorem 17, is not free of restrictive assumptions on the type of group and representations that we allow; the analysis of the hypothesis (§5.3) proves to be an interesting exercise in group theory. It raises pertinent questions – whose answers may be well known – concerning the nature of groups over discrete valuation rings. (For example: are there affine and flat group schemes over a DVR of mixed characteristic whose special fibre is trivial?)

In section 6 we study the natural formal deformation problem associated to liftings (or deformations) of D-modules; the mould is that of [38] and [24], while the main inspiration is the seminal work of Mazur [28] on deformations of Galois representations. The main product, Theorem 26, shows that the corresponding deformation functor or moduli problem (Definition 19) is homogeneous (Definition 20). Due to a celebrated theorem of Schlessinger [38, Thm. 2.11], homogeneity in the presence of finite dimensionality of tangent spaces means representability. But, as the tangent space of the deformation functor proposed at this point has no reason whatsoever to be of finite dimension, we will need to be more selective in the deformations allowed; this is the reason for section 7.

In section 7 we propose to study certain classes of deformations – called periodic (Definition 30) – of a D-module in positive characteristic. Using methods similar to those of section 6, we obtain the homogeneity (Definition 20) of the functor associated to this problem (Theorem 31). The introduction of these classes is motivated by the fact that the corresponding deformation functor has a finite dimensional tangent space in many cases, as will be seen in section 8. The idea behind the notion of periodic deformations is that the "solutions" of the deformed D-module will appear in a *fixed* extension of the base ring. Perhaps the mental image, suggested by the analytic picture, is that of a family of representations whose image lies in a fixed subgroup of some general linear group. (We are inclined to use the word isomonodromy, but this may not be helpful.)

In section 8 we carry the calculations of the tangent spaces to the functors introduced (see Definition 21) in sections 6 and 7. (The calculation of t_{Def} is completely standard and is included in the text for lack of a mild reference.) The principal result is Proposition 33, which hints that for a large class of differential Galois groups, the tangent space is finite dimensional; we also state Corollary 35 in order to encapsulate our findings.

The last section, section 9, is devoted to showing that the formal deformation theory proposed in section 7 is *isomorphic* to the formal deformation theory of the representation attached to the DGG of the D-module in positive characteristic (see Theorem 38). This is reminiscent of the well known fact from differential Geometry: deforming a flat connection is equivalent to deforming a representation of the topological fundamental group [15]. Clearly, there is a considerable difference here, which is the infancy of a theory for the fundamental group scheme, specially when it varies over a DVR. As an application, we obtain a result (Corollary 41) which allows us to say when two liftings to characteristic zero are isomorphic as D-modules.

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1.2. Conventions and notations

1.2.1. *The base ring.—* We fix Λ a complete discrete valuation ring. Its field of fractions, which will be denoted by K, is of characteristic zero; the residue field, which will be denoted by k, is algebraically closed and of characteristic $p > 0$; the uniformizer will be denoted by ϖ.