Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 139
Fascicule 2

2011

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 271-286

Bull. Soc. math. France
139 (2), 2011, p. 271–286

A CENTRAL LIMIT THEOREM FOR TWO-DIMENSIONAL RANDOM WALKS IN A CONE

BY RODOLPHE GARBIT

ABSTRACT. — We prove that a planar random walk with bounded increments and mean zero which is conditioned to stay in a cone converges weakly to the corresponding Brownian meander if and only if the tail distribution of the exit time from the cone is regularly varying. This condition is satisfied in many natural examples.

RÉSUMÉ (*Un théorème limite central pour des marches aléatoires dans des cônes du plan*)

Nous démontrons qu'une marche aléatoire dans le plan, centrée, à accroissements bornés, et conditionnée à rester dans un cône, converge en loi vers le méandre brownien correspondant si et seulement si la queue de la loi du temps de sortie du cône est à variation régulière. Cette condition est satisfaite dans de nombreux exemples naturels.

1. Introduction

1.1. Main result. — The aim of this paper is to underscore a natural necessary and sufficient condition for the weak convergence of a two-dimensional random walk conditioned to stay in a cone to the corresponding Brownian meander.

Texte reçu le 26 novembre 2009, accepté le 28 août 2010

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2000 Mathematics Subject Classification. — 60F17, 60G50, 60J05, 60J65.

Key words and phrases. — Conditioned random walks, Brownian motion, Brownian meander, cone, functional limit theorem, regularly varying sequences.

This work was partially supported by Agence Nationale de la Recherche Grant ANR-09-BLAN-0084601.

The condition only involves the asymptotic behavior of the tail distribution of the first exit time from the cone.

Let $(\xi_n)_{n\geq 1}$ be a sequence of independent and identically distributed random vectors of $\mathbb{R}^d$, $d \geq 1$, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We assume that the distribution of ξ_1 satisfies $\mathbb{E}(\xi_1) = 0$ and $Cov(\xi_1) = \sigma^2 I_d$, where $\sigma^2 > 0$ and I_d is the $d \times d$ identity matrix.

We form the random walk $S = (S_n)_{n\geq 1}$ by setting $S_n = \xi_1 + \cdots + \xi_n$, and for each $n \geq 1$, we define a normed and linearly interpolated version of S by

$$\mathscr{S}_n(t) = \frac{S_{[nt]}}{\sigma\sqrt{n}} + (nt - [nt])\frac{\xi_{[nt]+1}}{\sigma\sqrt{n}}, \quad t \geq 0,$$

where $[a]$ denotes the integer part of a.

The weak convergence of the process $\mathscr{S}_n = (\mathscr{S}_n(t), t \geq 0)$ as $n \to \infty$ to a standard Brownian motion is Donsker's theorem (see for example Theorem 10.1 of [1]).

We consider a linear cone $C \subset \mathbb{R}^d$ (*i.e.* $\lambda C = C$ for every $\lambda > 0$) with the following properties:

1. C is convex,
2. its interior C^o is non-empty,
3. $\mathbb{P}(\xi_1 \in C \setminus \{0\}) > 0$.

Such a cone is said to be *adapted* to the random walk. Note that the convexity of C ensures that its boundary ∂C is negligible with respect to Lebesgue measure (see for example [8]). The third condition ensures that the first step of the random walk is in C with positive probability. Since a convex cone is a semi-group, the event $\{\xi_1, \ldots, \xi_n \in C\}$ is a subset of $\{S_1, \ldots, S_n \in C\}$, so the latter has also a positive probability. For this purpose, one could simply require that $\mathbb{P}(\xi_1 \in C) > 0$, but our third condition also excludes the uninteresting cases where $\{S_1, S_2, \ldots, S_n \in C\} = \{S_1 = S_2 = \cdots = S_n = 0\}$ almost surely.

We consider the first exit time of the random walk from the cone defined by

$$T_C = \inf\{n \geq 1 : S_n \notin C\},$$

and wish to investigate the asymptotic distribution of $(S_1, \ldots, S_n)$ conditional on $\{T_C > n\}$ as $n \to \infty$.

We denote by $\mathscr{C}_1$ the space of all continuous functions $w : [0,1] \to \mathbb{R}^d$, endowed with the topology of the uniform convergence and the corresponding Borel σ-algebra. Weak convergence of probability measures on $\mathscr{C}_1$ will be denoted by the symbol $\Rightarrow$.

Let Q_n denote the distribution on $\mathscr{C}_1$ of the process $\mathscr{S}_n$ conditional on $\{T_C > n\}$, that is, for any Borel set B of $\mathscr{C}_1$,

$$Q_n(B) = \mathbb{P}(\mathscr{S}_n \in B | T_C > n).$$

Note that, since C is a convex cone, this is equivalent to conditioning $\mathscr{S}_n$ on $\{\tau_C(\mathscr{S}_n) > 1\}$, where

$$\tau_C(w) = \inf\{t > 0 : w(t) \notin C\}, \quad w \in \mathscr{C}_1.$$

We are interested in the weak convergence of the sequence of conditional distributions (Q_n). The one-dimensional case, where $C = [0, \infty)$, has been investigated in the 60's and the 70's by many authors. It was Spitzer [11] who first announced a central limit theorem for the random walk conditioned to stay positive:

$$Q_n(w(1) \leq x) \to 1 - \exp(-x^2/2), \quad x \geq 0.$$

But, apparently, he never published the proof. Note that the limit is the Rayleigh distribution. A first proof of the weak convergence of Q_n was given by Iglehart in [7] under the assumptions $\mathbb{E}(|\xi_i|^3) < \infty$ and ξ_i nonlattice or integer valued with span 1. The limit is found to be the distribution of Brownian meander. Then Bolthausen proved in [3] that these extra assumptions were superfluous. For the reader who may not be familiar with the Brownian meander, we will use a theorem of Durrett, Iglehart and Miller [4] as a definition. Let W^x be the distribution of the standard Brownian motion started at x. For any $x > 0$, we denote by M^x the distribution W^x conditional on $\{\tau_C > 1\}$, that is

$$M^x(B) = W^x(B | \tau_C > 1)$$

for any Borel set B of $\mathscr{C}_1$. Here, the definition of conditional probabilities is elementary since $W^x(\tau_C > 1)$ is positive for all $x > 0$. The distribution M of Brownian meander is the weak limit of M^x as $x \to 0^+$ (see [4], Theorem 2.1). Note that the existence of a limit is not straightforward since $W^0(\tau_C > 1) = 0$. But in a sense, the Brownian meander is a Brownian motion started at 0 and conditioned to stay positive for a unit of time. The Brownian meander can alternatively be obtained by some path transformations of Brownian motion. Namely, it is the first positive excursion of Brownian motion with a lifetime greater than 1; it is also the absolute value of the rescaled section of Brownian motion observed on the interval $[h, 1]$, where h is its last zero before $t = 1$ (see [3] and [4]).

With this in mind, the weak convergence of Q_n to M can be stated in the following imprecise but intuitive way: the random walk conditioned to stay positive converges to a Brownian motion conditioned to stay positive.

We now turn to the two-dimensional case. If Q_n does converge weakly, then its limit should naturally be the distribution of a Brownian motion conditioned to stay in the cone C for a unit of time. Such a process can be defined as the weak limit of conditioned Brownian motion in the same way as Brownian meander. As above, for $x \in C^o$, let M^x be the distribution of Brownian motion started at x and conditioned to stay in C for a unit of time. The following theorem is due to Shimura [9] and has been extended in [6] to any dimension $d \geq 2$ for smooth cones.

THEOREM 1.1 ([9], Theorem 2). — *As $x \in C^o \to 0$, the distribution M^x converges weakly to a limit M.*

The limit distribution M in this theorem will be referred to as the distribution of the Brownian meander (of the cone C). We will give more details about M in Section 2.

We now come to the main result of the present paper. We recall that a sequence (u_n) of positive numbers is *regularly varying* if it can be written as $u_n = n^{-\alpha} l_n$, where $\alpha \in \mathbb{R}$ and (l_n) is slowly varying, *i.e.* $\lim_n l_{[nt]}/l_n = 1$ for all $t > 0$ (see for example [2]). The exponent α is unique and called the index of regular variation. A non-increasing sequence of positive numbers (u_n) will be called *dominatedly varying*(1) if $\limsup_n u_{[nt]}/u_n$ is finite for all $t \in (0, 1]$.

THEOREM 1.2. — *Assume that the two-dimensional random walk has bounded increments. Then, the sequence of conditional distributions (Q_n) converges weakly to the Brownian meander if and only if $\mathbb{P}(T_C > n)$ is dominatedly varying. In that case, $\mathbb{P}(T_C > n)$ is regularly varying with index $\pi/(2\beta)$, where β is the angle of the cone.*

The assumption of bounded increments is only used in the proof of the tightness of (Q_n) which is taken from the paper [10] of Shimura. We will discuss some extensions to the case where the increments are not bounded in Section 3.1. However, the rest of the proof of Theorem 1.2, which consists in a study of the (eventual) limit points of the sequence (Q_n), is completely independant of the assumption of bounded increments. Thus, we could have stated a more general (but not very useful) theorem by simply assuming that (Q_n) is tight. In order to avoid any confusion, the reader is advised that in any of the lemmas, propositions or theorems of this paper, the random walk (S_n) is not assumed to have bounded increments unless it is written explicitly.

Our Theorem 1.2 can be regarded as an extension of a previous result due to Shimura ([10], Theorem 1). Indeed, he proved that $Q_n \Rightarrow M$ if the distribution of the increments satisfies the following condition: there exists an orthogonal basis $\{\vec{u}, \vec{v}\}$ of $\mathbb{R}^2$ with $\vec{v} \in C^o$ such that $\mathbb{E}(V|U) = 0$, where (U, V) denotes the coordinates of ξ_1 in the new basis. But this condition does not seem to be

(1) This is strictly weaker than regular variation. For example, since $\prod_n (1+1/n)$ is divergent, it is possible to construct a sequence of numbers $1 \leq c_n \leq 2$ such that :(i) for all $n \geq 1$, $c_{n+1} \leq (1 + 1/n)c_n$, and (ii) for all $\epsilon > 0$, there exist infinitely many n such that $c_n \geq 2 - \epsilon$ and $c_{n+1} = 1$. Then, the sequence $u_n = c_n/n$ is non-increasing and dominatedly varying, but not regularly varying since $\liminf u_{n+1}/u_n \leq 1/2$ is not equal to 1 as it should be (see [13] for a very nice proof of this).