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HILBERT SCHEMES AND STABLE PAIRS: GIT AND DERIVED CATEGORY WALL CROSSINGS

by Jacopo Stoppa & Richard P. Thomas

Abstract. — We show that the Hilbert scheme of curves and Le Potier's moduli space of stable pairs with one dimensional support have a common GIT construction. The two spaces correspond to chambers on either side of a wall in the space of GIT linearisations.

We explain why this is not enough to prove the "DT/PT wall crossing conjecture" relating the invariants derived from these moduli spaces when the underlying variety is a 3-fold. We then give a gentle introduction to a small part of Joyce's theory for such wall crossings, and use it to give a short proof of an identity relating the Euler characteristics of these moduli spaces.

When the 3-fold is Calabi-Yau the identity is the Euler-characteristic analogue of the DT/PT wall crossing conjecture, but for general 3-folds it is something different, as we discuss.

Résumé (*Schémas de Hilbert et paires stables : GIT et croisements de murs de catégories dérivées*)

Nous montrons que le schéma de Hilbert de courbes et l'espace de modules de Le Potier de paires stables à support à une dimension, ont une construction GIT commune. Les deux espaces correspondent aux chambres de par et d'autre d'un mur dans l'espace de linéarisations GIT.

Nous expliquons pourquoi cela ne suffit pas pour prouver la « conjecture de croisement de murs DT/PT » qui relie les invariants dérivés de ces espaces de modules quand la variété sous-jacente est un 3-fold. Nous donnons, ensuite, une introduction simple à une petite partie de la théorie de Joyce sur les croisements de murs de ce type,

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et nous nous en servons pour donner une brève démonstration d'une identité reliant les caractéristiques d'Euler de ces espaces de modules.

Quand le 3-fold est de type Calabi-Yau, l'identité est le pendant, pour la caractéristique d'Euler, de la conjecture de croisement de murs DT/PT, mais dans le cas général elle s'avère être différente de celle-ci, comme nous l'expliquons.

1. Introduction

This paper is motivated by the conjectural equivalence between two curve counting theories on smooth complex projective threefolds X: the one studied in [17] and the stable pairs of [21].

MNOP and stable pairs invariants are sheaf-theoretic analogues of Gromov-Witten invariants, sometimes called DT and PT invariants respectively. The space of stable maps to X is replaced by suitable moduli spaces of sheaves supported on curves in X.

Fix $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$. In MNOP theory we integrate suitable classes against the virtual fundamental class of the Hilbert scheme $I_n(X, \beta)$ of subschemes Z of X in the class $[Z] = \beta$ with holomorphic Euler characteristic $\chi(\mathcal{O}_Z) = n$. The virtual fundamental class comes from thinking of $I_n(X, \beta)$ as a moduli space of sheaves of trivial determinant – namely the ideal sheaves $\mathscr{I}_Z$ with Chern character

$$(1.1) \qquad \Big(1, 0, -\beta, -n + \frac{\beta.c_1(X)}{2}\Big) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X).$$

For stable pair theory we work instead with *stable pairs* (F, s): F is a *pure* sheaf on X with Chern character $(0, 0, \beta, -n + \beta.c_1(X)/2)$, and $s\colon \mathcal{O}_X \to F$ is a section with 0-dimensional cokernel. A special case of the work of Le Potier [14] constructs the fine moduli space $P_n(X, \beta)$ as a projective scheme. The virtual fundamental class comes from thinking [21] of $P_n(X, \beta)$ as a moduli space of objects of the derived category of coherent sheaves on X (with trivial determinant) – namely the complexes $I^\bullet := \{\mathcal{O}_X \to F\}$ with Chern character (1.1).

Roughly speaking, we think of $I_n(X, \beta)$ as parameterising pure curves plus points (free and embedded) on X. Any $Z \in I_n(X, \beta)$ contains a maximal Cohen-Macaulay curve $C \subseteq Z$ (the pure curve: recall that Cohen-Macaulay means no embedded points) such that the kernel of $\mathcal{O}_Z \to \mathcal{O}_C$ is 0-dimensional (the points). Equally loosely we think of stable pairs as parameterising Cohen-Macaulay curves (the support of the sheaf F) and free points *on the curve* (the cokernel of the section s).

Over the Zariski-open subset of Cohen-Macaulay curves C with no free or embedded points, the moduli spaces $I_n(X, \beta)$ and $P_n(X, \beta)$ are isomorphic: the stable pair $\mathcal{O}_X \to \mathcal{O}_C$ determines and is determined by the kernel ideal sheaf $\mathscr{I}_C$. Indeed $I^\bullet$ is quasi-isomorphic to $\mathscr{I}_C$.

When $\omega_X \cong \mathcal{O}_X$, i.e. X is a Calabi-Yau threefold, MNOP and stable pair invariants take a particularly simple form. Then the virtual dimension is zero and we get invariants by taking the length of the 0-dimensional virtual cycle:

$$I^{\mathrm{vir}}_{m,\beta} = \int_{[I_m(X,\beta)]^{\mathrm{vir}}} 1,$$

and

$$P^{\mathrm{vir}}_{m,\beta} = \int_{[P_m(X,\beta)]^{\mathrm{vir}}} 1.$$

In this case the deformation-obstruction theories [24, 21, 9] used to define the virtual cycles are *self dual* in the sense of [2]. This implies that $I^{\mathrm{vir}}_{m,\beta}$, $P^{\mathrm{vir}}_{m,\beta}$ are in fact weighted Euler characteristics:

$$I^{\mathrm{vir}}_{m,\beta} = e(I_m(X,\beta),\chi^B), \qquad P^{\mathrm{vir}}_{m,\beta} = e(P_m(X,\beta),\chi^B).$$

Here the weighting function is Behrend's integer-valued constructible function χ^B [2], which assigns to each point of the moduli space the multiplicity with which it contributes to the invariants. At smooth points of the moduli space, $\chi^B \equiv (-1)^{\dim}$.

We can also form their generating series

$$Z^{I,\mathrm{vir}}_\beta(X)(t) := \sum_{m\in\mathbb{Z}} I^{\mathrm{vir}}_{m,\beta} t^m \qquad \text{and} \qquad Z^{P,\mathrm{vir}}_\beta(X)(t) := \sum_{m\in\mathbb{Z}} P^{\mathrm{vir}}_{m,\beta} t^m.$$

The conjectural equivalence between the MNOP and stable pair invariants in the Calabi-Yau case is then the following.

Conjecture 1.2. — [21] *For X a Calabi-Yau threefold,*

$$Z^{P,\mathrm{vir}}_\beta(X) = \frac{Z^{I,\mathrm{vir}}_\beta(X)}{Z^{I,\mathrm{vir}}_0(X)}\,.$$

Equivalently, for each $m \in \mathbb{Z}$ we have the following identity (where the right hand side is a finite sum):

$$I^{\mathrm{vir}}_{m,\beta} = P^{\mathrm{vir}}_{m,\beta} + I^{\mathrm{vir}}_{1,0}\cdot P^{\mathrm{vir}}_{m-1,\beta} + I^{\mathrm{vir}}_{2,0}\cdot P^{\mathrm{vir}}_{m-2,\beta} + \cdots\,. \tag{1.3}$$

Here $Z^{I,\mathrm{vir}}_0(X)$ is the generating series of virtual counts of zero dimensional subschemes of X. By [17, 3, 15, 16] it is in fact

$$Z^{I,\mathrm{vir}}_0(X)(t) = M(-t)^{e(X)},$$

where $M(t)$ is the MacMahon function, the generating function for 3-dimensional partitions.

Using Kontsevich-Soibelman's identities for χ^B [13], now proved in some cases [12], it should now be possible to extend what follows to the weighted

Euler characteristics $I^{\text{vir}}_{m,\beta}$ and $P^{\text{vir}}_{m,\beta}$. But in this paper we content ourselves with working with the unweighted Euler characteristics

$$I_{m,\beta} := e(I_m(X,\beta)) \qquad \text{and} \qquad P_{m,\beta} := e(P_m(X,\beta)),$$

which are not deformation invariant. Form their generating series

$$Z^I_\beta(X)(t) := \sum_{m\in\mathbb{Z}} I_{m,\beta}t^m \qquad \text{and} \qquad Z^P_\beta(X)(t) := \sum_{m\in\mathbb{Z}} P_{m,\beta}t^m.$$

In Sections 4 and 5 we will give two different proofs of the following topological analogue of Conjecture 1.2 (first proved by Toda [26] in the Calabi-Yau case, as discussed below).

Theorem 1.4. — *Let X be a smooth projective threefold. Then*

$$Z^P_\beta(X) = \frac{Z^I_\beta(X)}{Z^I_0(X)}\,.$$

Equivalently for each $m \in \mathbb{Z}$ we have the following identity (where the right hand side is a finite sum):

$$I_{m,\beta} \;=\; P_{m,\beta} \;+\; I_{1,0}\cdot P_{m-1,\beta} \;+\; I_{2,0}\cdot P_{m-2,\beta} \;+\; \cdots\,.$$

Here the $I_{k,0} = e(\operatorname{Hilb}^k X)$ are the Euler characteristics of the Hilbert schemes of points on X, and $Z^I_0(X)$ is their generating series. By [5] this is

$$Z^I_0(X) = M(t)^{e(X)}.$$

In fact we prove a little more. Fixing a Cohen-Macaulay C in class β, define $I_{n,C}$ to be the Euler characteristic of the subset of $I_n(X,\beta)$ consisting of subschemes whose underlying Cohen-Macaulay curve is C (this is naturally a projective scheme, see below). Similarly let $P_{n,C}$ be the Euler characteristic of the subset (in fact projective scheme) of $P_n(X,\beta)$ of pairs supported on C.

Theorem 1.5. — *Let $C \subset X$ be a Cohen-Macaulay curve in a smooth projective threefold. Then*

$$I_{n,C} = P_{n,C} + e(X)P_{n-1,C} + e(\operatorname{Hilb}^2 X)P_{n-2,C} + \cdots + e(\operatorname{Hilb}^n X)P_{0,C}\,.$$

We explain in Section 4.2 how to deduce Theorem 1.4 from this by "integrating" over the space of Cohen-Macaulay curves C.

Naively one should think of the above identities as reflecting the decomposition of $I_n(X,\beta)$ into a union of the subset of pure Cohen-Macaulay curves with no free or embedded points, the subset with one free or embedded point, the subset with two points, etc. Birationally such a decomposition is given by (1.6) below.