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STRATIFIED WHITNEY JETS AND TEMPERED ULTRADISTRIBUTIONS ON THE SUBANALYTIC SITE

BY N. HONDA & G. MORANDO

ABSTRACT. — In this paper we introduce the sheaf of stratified Whitney jets of Gevrey order on the subanalytic site relative to a real analytic manifold X. Then, we define stratified ultradistributions of Beurling and Roumieu type on X. In the end, by means of stratified ultradistributions, we define tempered-stratified ultradistributions and we prove two results. First, if X is a real surface, the tempered-stratified ultradistributions define a sheaf on the subanalytic site relative to X. Second, the tempered-stratified ultradistributions on the complementary of a 1-regular closed subset of X coincide with the sections of the presheaf of tempered ultradistributions.

RÉSUMÉ (*Jets stratifiés de Whitney et ultradistributions tempérées sur le site sous-analytique*)

Dans cet article nous introduisons le faisceau des jets de Whitney d'ordre de Gevrey sur le site sous-analytique relatif à une variété analytique réelle X. Ensuite, nous définissons les ultradistributions tempérées sur X de type Beurling et Roumieu. Enfin, à travers les ultradistributions stratifiées, nous définissons les ultradistributions tempérées-stratifiées et nous démontrons les deux résultats suivants : (a) si X est une

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surface réelle, les ultradistributions tempérées-stratifiées définissent un faisceau sur le site sous-analytique relatif à X, et (b) les ultradistributions tempérées-stratifiées sur le complémentaire d'un sous-ensemble fermé 1-régulier de X, coïncident avec les sections du préfaisceau des ultradistributions tempérées.

Introduction

One of the aims of the present article is to define tempered ultradistributions of Beurling and Roumieu class of order $s > 1$ and Whitney jets with growth conditions as sheaves on the subanalytic site relative to a real analytic manifold X. As growth conditions are not of local nature, functional spaces defined on open subsets of X, as tempered distributions, Whitney $\mathscr{C}^\infty$-functions or holomorphic functions with polynomial growth at the boundary do not glue on arbitrary coverings. In particular, such spaces do not define sheaves on the usual topology of an analytic manifold. We recall the approach set by S. Łojasiewicz ([12]) later reinterpreted and generalized in the works of M. Kashiwara and P. Schapira (see [4], [6] and [7]). They defined tempered distributions and Whitney $\mathscr{C}^\infty$-functions as sheaves on the subanalytic site, X_{sa}, relative to a real analytic manifold X. The open sets of X_{sa} are the relatively compact subanalytic open subsets of X and the coverings are those admitting finite refinements. The use of these objects in the study of linear ordinary differential equations gave interesting results (see [14]). Let us mention that function spaces with growth conditions, such as holomorphic functions on the complex plane with moderate or Gevrey growth or asymptotic expansion at the origin, are treated as sheaves on the real blow up at the origin by B. Malgrange in [13] and many other authors elsewhere in litterature. Such function spaces are used in a systematic way in the study of linear ordinary differential equations. Some of these sheaves on the real blow up at the origin can be obtained by specializing their subanalytic generalization (see [15]).

Among the motivations of this paper there is the fact that the naive definition of tempered ultradistributions, mimicking that of tempered distributions (see [4]), does not give a sheaf on the subanalytic site, as explained in Section 1.3. Let us recall that tempered ultradistributions on an open set U in X are defined as global sections of ultradistributions modulo ultradistributions with support on $X \setminus U$. This latter space is the dual of Whitney jets with Gevrey like growth conditions on $X \setminus U$. In this paper, we relax the condition on Whitney jets with Gevrey like growth conditions by introducing the stratified Whitney jets on a real analytic manifold X. We prove decomposition and gluing properties for stratified Whitney jets on finitely many subanalytic subsets of X (Lemma 2.2.4). Then we study the dual of stratified Whitney jets on a closed set $Z \subset X$, the space of stratified ultradistributions on Z. This latter space is a subspace of ultradistributions with support in Z. We study the decomposability of stratified ultradistributions on arbitrary finitely many subanalytic closed sets (Corollary 2.4.5 and Corollary 3.1.5). Then we define tempered-stratified ultradistributions on U as global ultradistributions modulo stratified ultradistributions on $X \setminus U$. We prove that, when X has dimension 2, tempered-stratified ultradistributions define a sheaf on X_{sa}. Further, we prove that, if $X \setminus U$ satisfies a regularity condition, tempered-stratified ultradistributions on U coincide with classical tempered ultradistributions on U (Theorem 3.2.1). We conclude by proving that tempered-stratified ultradistributions and other spaces of ultradistributions similarly defined do not give rise to sheaves on X_{sa}, if X has dimension > 2.

Similar results on the decomposability of ultradistributions were obtained by J.-M. Kantor ([3]) and by A. Lambert ([11]). Their approach is quite different from our. Indeed, given $s > 1$, they find a family $\mathcal{T}_s$ of subanalytic closed sets depending on s such that ultradistributions of class s decompose on sets in $\mathcal{T}_s$. The family $\mathcal{T}_s$ is not closed under intersections hence it is not possible to define a Grothendieck topology and a notion of sheaf starting from it.

In the end, let us recall that ultradistributions and growth conditions of Gevrey type turned out to be very useful in the functorial study of linear diffential equations, being strictly linked to the irregularity of equations. Let us cite, for example, [2] and [19] for some applications of ultradistributions in the study of systems of linear differential equations. In the present article we do not use tempered-stratified ultradistributions to study systems of linear differential equations, postponing this problem to future investigations. Throughout the paper, we just limit to point out if the sheaves we define give rise to sheaves of modules over the ring of linear differential operators with analytic coefficients.

The paper is organized as follows. We start Section 1 by recalling the basic properties of Whitney jets with growth conditions. Then, mimicking [4], we define the presheaf of tempered ultradistributions and we recall a condition, due to H. Komatsu, for a continuous function to extend to the whole space as an ultradistribution. In the end of the section, we prove that tempered ultradistributions do not glue on finitely many subanalytic open subsets of $\mathbb{R}^2$.

In Section 2 we start by recalling some definitions and basic results on subanalytic sets and the subanalytic site relative to a real analytic manifold X. Then, we define the space of stratified Whitney jets with Gevrey growth conditions and we prove that they give rise to a sheaf on the subanalytic site relative to X. Then, we introduce the space of stratified ultradistributions on X and we prove that this space is dual to stratified Whitney jets. In the end of the section, from the gluing property of stratified Whitney jets, we obtain a decomposition property for stratified ultradistributions.

In Section 3, given a real analytic manifold X, we define tempered-stratified ultradistributions on a subanalytic open set $U \subset X$ which is a subspace of tempered ultradistributions on U. Then, we prove two results. The first states that, if $\dim X = 2$, tempered-stratified ultradistributions define a sheaf on the subanalytic site relative to X. The second states that if $X \setminus U$ satisfies a regularity condition, then tempered-stratified ultradistributions on U coincide with tempered ultradistributions on U.

In Appendix A we prove a result of density for stratified Whitney jets in the space of Whitney jets. Such result is needed in Section 2, we prove it in the Appendix as the proof is rather long and technical.

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1. Notations and review on Whitney jets and ultradistributions

In this paper, we assume that a real analytic manifold is countable at infinity.

1.1. Whitney jets with Gevrey conditions. — Let X be a real analytic manifold. We denote by $\mathrm{Mod}(\mathbb{C}_X)$ the category of sheaves on X with values in $\mathbb{C}$-vector spaces, and by $\mathscr{C}^\infty$ the sheaf of infinitely differentiable functions on X. We denote by $\pi_k : J^k \to X$ ($k \in \mathbb{Z}_{\geq 0}$) the vector bundle associated with k-th jets over X. For any non-negative integers $k_1 \geq k_2$, the morphism of vector bundles $j^{k_2,k_1} : J^{k_1} \to J^{k_2}$ is defined by the canonical projection from k_1-th jets to k_2-th jets.

Let A be a locally closed subset in X, and $J^k(A)$ designates the set of continuous sections of the vector bundle J^k over A. We denote by $j^k_X : \mathscr{C}^\infty(X) \to J^k(X)$ the canonical jets extension morphism, and for any locally closed spaces $A \subset B$, we designate by $j^k_{A,B} : J^k(B) \to J^k(A)$ the natural restriction map from sections over B to those over A. Composing j^k_X and $j^k_{A,X}$ we have the canonical morphism

$$j^k_A = j^k_{A,X} \circ j^k_X : \mathscr{C}^\infty(X) \to J^k(A).$$

The morphism of vector bundles j^{k_2,k_1} ($k_1 \geq k_2$) induces the map

$$j^{k_2,k_1}_A : J^{k_1}(A) \to J^{k_2}(A),$$