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GEOMETRIC STABILITY OF THE COTANGENT BUNDLE AND THE UNIVERSAL COVER OF A PROJECTIVE MANIFOLD

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with an appendix by Matei Toma

Abstract. — We first prove a strengthening of Miyaoka's generic semi-positivity theorem: the quotients of the tensor powers of the cotangent bundle of a non-uniruled complex projective manifold X have a pseudo-effective (instead of generically nef) determinant. A first consequence is that X is of general type if its cotangent bundle contains a subsheaf with 'big' determinant. Among other applications, we deduce that if the universal cover of X is not covered by compact positive-dimensional analytic subsets, then X is of general type if $\chi(O_X) \neq 0$. We finally show that if L is a numerically trivial line bundle on X, and if $K_X + L$ is $\mathbb{Q}$-effective, then so is K_X itself. The proof of this result rests on Simpson's work on jumping loci of numerically trivial line bundles, and Viehweg's cyclic covers. This last result is central, and has been recently extended, using the very same ingredients, to the case of log-canonical pairs.

Résumé (*Stabilité géométrique du fibré cotangent et du recouvrement universel d'une variété projective*)

Nous établissons tout d'abord un renforcement du théorème de semi-positivité de Miyaoka: le déterminant de tout quotient de toute puissance tensorielle du fibré cotangent d'une variété projective X non-uniréglée est pseudo-effectif (au lieu de: génériquement nef). Une première conséquence est que X est de type général si son fibré

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cotangent a un sous-faisceau dont le déterminant est ‘big’. Parmi diverses applications, nous montrons que si le revêtement universel de X n’est pas recouvert par des sous-ensembles analytiques compacts de dimension strictement positive, alors X est de type général si $\chi(O_X) \neq 0$.Nous montrons enfin que K_X est $\mathbb{Q}$-effectif si $K_X + L$ l’est, pour un fibré en droites numériqiuement effectif L sur X. La démonstration de ce résultat central repose sur les travaux de C. Simpson sur les lieux de Green-Lazarsfeld, et sur les revêtements cycliques de Viehweg. Ce résultat a été récemment étendu aux paires ’Log-canoniques’ en utilisant les mêmes ingrédients.

Introduction

The aim of the present paper is to investigate birational positivity properties of the cotangent bundle of complex projective manifolds.

Our first result is the following sharpening of Miyaoka’s uniruledness criterion:

THEOREM 0.1. — *Let X be a projective manifold, $(\Omega^1_X)^{\otimes m} \to \mathcal{Q}$ a torsion free coherent quotient for some $m \in \mathbb{N}$. Then* $\det \mathcal{Q}$ *is pseudo-effective if X is not uniruled.*

Miyaoka’s theorem asserts that the cotangent bundle of a projective manifold is “generically nef” unless the manifold is uniruled. A vector bundle E is *generically nef* if $E|C$ is nef on the general curve cut out by very ample linear systems of sufficiently high degree. A line bundle L is *pseudo-effective* if $c_1(L)$ lies in the closure of the Kähler cone. To sharpen generic nefness to pseudo-effectivity in the theorem , we use the characterization [2] of pseudo-effective line bundles by moving curves which are images of very ample curves above by birational morphisms. Our proof here is not entirely algebro-geometric (Mehta-Ramanathan no longer applies), and rests on analytic methods (see the appendix due to M. Toma).

A first consequence is:

THEOREM 0.2. — *Let X be a projective manifold. Suppose that Ω^p_X contains for some p a subsheaf whose determinant is big (i.e., has Kodaira dimension $n = \dim X$). Then K_X is big, i.e., $\kappa(X) = n$.*

This uniruledness criterion has also other applications, e.g. one can prove that a variety admitting a section in a tensor power of the tangent bundle with a zero, must be uniruled.

Theorem 0.2 is actually a piece in a larger framework. To explain this, we consider subsheaves $\mathcal{F} \subset \Omega^p_X$ for some $p > 0$. Then one can form $\kappa(\det \mathcal{F})$ and take the supremum over all $\mathscr{F}$. This gives a refined Kodaira dimension $\kappa^+(X)$, introduced in [3]. Conjecturally

$$\kappa^+(X) = \kappa(X) \tag{$*$}$$

unless X is uniruled. Theorem 0.2 is nothing but this conjecture in case $\kappa^+(X) = \dim X$.

We shall prove the conjecture $(*)$ in several other cases. It is actually a consequence of the following more general conjecture, which moreover deals only with line bundles:

Conjecture. — *Suppose X is a projective manifold, and suppose a decomposition*

$$NK_X = A + B$$

with some positive integer N, an effective divisor A (one may assume A spanned) and a pseudo-effective line bundle B. Then

$$\kappa(X) \geq \kappa(A).$$

The special case $A = \mathscr{O}_X$ implies that $\kappa(X) \geq 0$ if X is not uniruled, using the preceeding result, and the pseudo-effectiveness of K_X when X is not uniruled ([2]).

In another direction we establish the special case in which B is numerically trivial:

Theorem 0.3. — *Let X be a projective complex manifold, and $L \in \mathrm{Pic}(X)$ be numerically trivial. Then:*

1. $\kappa(X, K_X + L) \leq \kappa(X)$.
2. *If $\kappa(X) = 0$, and if $\kappa(X, K_X + L) = \kappa(X)$, then L is a torsion element in the group* $\mathrm{Pic}^0(X)$.

In particular, if mK_X is numerically equivalent to an effective divisor, then $\kappa(X) \geq 0$.

This result permits, in particular, to handle numerically trivial line bundles in the study of the conjecture $C_{n,m}$ on irregular manifolds.

Another application of Theorem 0.2 is to the study of universal covers $\tilde{X}$ of complex projective n-dimensional manifolds X. The Shafarevich conjecture asserts that $\tilde{X}$ is holomorphically convex, i.e., admits a proper holomorphic map onto a Stein space. There are two extremal cases:

- either $\tilde{X}$ is compact and so $\pi_1(X)$ is finite or
- $\tilde{X}$ is a modification of a Stein space, hence through the general point of $\tilde{X}$ there is no positive-dimensional compact subvariety.

This latter case happens in particular for X a modification of an Abelian variety or a quotient of a bounded domain. It is conjectured (see [13], and [5] for the Kähler case) that X should then admit a holomorphic submersion onto a variety of general type with Abelian varieties as fibres, after a suitable finite étale cover and birational modification. This follows up to dimension 3 from the solutions of the conjectures of the Minimal Model Program. We prove here a special case and a weaker statement in every dimension:

THEOREM 0.4. — *Let X be a normal n-dimensional projective variety with at most rational singularities.*

(1) *Suppose that the universal cover of X is not covered by its positive-dimensional compact subvarieties. Then X is of general type if $\chi(\mathcal{O}_X) \neq 0$.*

(2) *If X has at most terminal singularities and $\tilde{X}$ does not contain any compact subvariety of positive dimension (eg. X is Stein), then either K_X is ample, or K_X is nef, $K_X^n = 0$, and $\chi(\mathcal{O}_X) = 0$.*

This theorem is deduced from Theorem 0.2 above via the comparison theorem [3], which relates the geometric positivity of subsheaves in the cotangent bundle to the geometry of $\tilde{X}$.

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1. Uniruledness Criteria

Our main tool which is of independent interest, is a generalisation 1.4 of Miyaoka's uniruledness Criterion 1.2, which we recall first.

DEFINITION 1.1. — *Let X be a complex projective n-dimensional manifold. A vector bundle E over X is generically nef, if for all ample line bundles $H_1, \dots, H_{n-1}$, for all m_i sufficiently large and for general curves C cut out by $m_1H_1, \dots, m_{n-1}H_{n-1}$, the bundle $E|C$ is nef.*

Miyaoka's criterion [17], with a short proof in [23], is now the following

THEOREM 1.2. — *The cotangent bundle of a projective manifold is generically nef if X is not uniruled.*