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AN *abcd* THEOREM OVER FUNCTION FIELDS AND APPLICATIONS

by Pietro Corvaja & Umberto Zannier

Abstract. — We provide a lower bound for the number of distinct zeros of a sum $1+u+v$ for two rational functions u,v, in term of the degree of u,v, which is sharp whenever u,v have few distinct zeros and poles compared to their degree. This sharpens the "*abcd*-theorem" of Brownawell-Masser and Voloch in some cases which are sufficient to obtain new finiteness results on diophantine equations over function fields. For instance, we show that the Fermat-type surface $x^a+y^a+z^c=1$ contains only finitely many rational or elliptic curves, provided $a \geq 10^4$ and $c \geq 2$; this provides special cases of a known conjecture of Bogomolov.

Résumé (*Un théorème abcd sur les corps de fonctions et applications*)

Nous démontrons une minoration pour le nombre de zéros distincts d'une somme $1+u+v$, u,v étant deux fonctions rationnelles, en fonction du degré de u et v; cette minoration est forte si le nombre de zéros et poles de u,v est suffisament petit par rapport à leur degré. Dans certains cas, on obtient une amélioration de l'inégalité de Voloch et Brownawell-Masser, qui entraîne des nouveaux résultats de finitude sur les équations diophantiennes sur les corps de fonctions.

Par exemple, nous démontrons que la surface de type Fermat définie par l'équation $x^a+y^a+z^c=1$ ne contient qu'un nombre fini de courbes rationnelles ou elliptiques, dès que $a \geq 10^4$ et $c \geq 2$. Ce résultat constitue un cas particulier d'une célèbre conjecture de Bogomolov.

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1. Introduction

We shall be interested in diophantine equations over function fields, involving S-units. We start by recalling a few definitions. Let κ be an algebraically closed field of characteristic zero, $\mathscr{C}$ a smooth complete curve of genus g defined over κ and S be a finite set of points of $\mathscr{C}$. By S-unit we mean a rational function $u \in \kappa(\mathscr{C})$ having all its poles and zeros on S. The group of S-units will be denoted by $\mathscr{O}_S^*$. Let χ be the Euler characteristic of $\mathscr{C} \setminus S$, i.e.

$$\chi = 2g - 2 + \sharp(S).$$

In the sequel we shall suppose $\sharp(S) \geq 2$, otherwise there would be no nonconstant S-units; note that this yields $\chi \geq 0$. We shall use the following notion of height relative to the function field $\kappa(\mathscr{C})$: if $(x_0 : \cdots : x_n) \in \mathbf{P}_n(\kappa(\mathscr{C}))$, we put

$$H(x_0 : \cdots : x_n) = -\sum_{\nu \in \mathscr{C}} \min\{\nu(x_0), \ldots, \nu(x_n)\},$$

where, for each $\nu \in \mathscr{C}(\kappa)$, we denote by the same letter $\nu : \kappa(\mathscr{C})^* \to \mathbf{Z}$ the order function corresponding to such point. So $H(1 : x)$ is the degree of the function x, which will also be written as $H(x)$.

Let u, v be S-units. We shall put

$$z := u + v + 1. \tag{1.1}$$

Whenever z is assumed to be an S-unit, (1.1) is the S-unit equation in three variables; the case of two variables gives rise to the so called *abc* theorem of Mason and Stothers, stating that *if $z = 0$ in (1.1) and u, v are not both constant, we have $H(1 : u : v) \leq \chi$.* This is best possible.

In (1.1), to avoid consideration of "trivial" cases, we shall assume that no subsum on the right vanishes. Under this condition, we have the inequalities of Brownawell-Masser, generalising the *abc* theorem; the case of four terms a, b, c, d implies for (1.1) that $H(1 : u : v : z) \leq 3\chi$, where the coefficient 3 replaces the previous coefficient 1. The constant 3 is sharp, in view of an example of Browkin-Brzezinski: $\kappa(\mathscr{C}) = \kappa(t)$, $u = -t^3, v = (t-1)^3, z = 3t(t-1)$. However, one expects improvements on the coefficient 3 under supplementary conditions; in fact a conjecture of Vojta predicts an estimate $H(1 : u : v : z) \leq (1 + \epsilon)\chi$ for every $\epsilon > 0$, provided the point $(1 : u : v)$ does not lie on a curve on $\mathbf{P}_2$, depending only on ϵ. See the discussion at §14.5.26 of [4].

Any improvement on the coefficient 3 may be crucial for applications to diophantine equations over function fields: for instance the equation $y^2 = 1 + u + v$ in S-units u, v and S-integer y escapes from the Brownawell-Masser estimates (which cover e.g. $y^3 = 1+u+v$), but could be treated by the analogue inequality with any coefficient < 2.

To improve on the methods of the paper [5] seems to lead to delicate problems. In the paper [6], by means of different methods, among other things we succeded to treat completely the said equation $y^2 = 1 + u + v$. A crucial ingredient was a bound for the gcd of $u - 1$, $v - 1$, for S-units u, v, which we recall below as Theorem CZ.

In the present paper, we develop certain applications of those results and methods especially to present a kind of general "*abcd* theorem" and some corollaries. Roughly speaking, we obtain the coefficient $1 + \epsilon$, whenever two of the S-units are multiplicatively independent modulo constants and the set of their zeros and poles has cardinality $< \delta\sharp(S)$, for a suitably small function $\delta = \delta(\epsilon, g)$ (see the Corollary to Theorem 1.1). One can easily show that in fact the same estimate holds also if, say, u, v are multiplicatively dependent modulo constants, unless they satisfy a multiplicative dependence relation of degree bounded in terms of ϵ. In other words, we obtain the estimate of Vojta's conjecture, although under a further condition, on the number of zeros of u, v.

This will be done in Theorem 1.1. Then we shall present some other applications, where we preferred simplicity to generality, to better illustrate the methods.

A first application concerns Fermat-type equations of the shape $x^a + y^b + 1 = z^c$, to be solved in non constant polynomials or rational functions $x, y, z \in \kappa(t)$. The quoted inequalities of Brownawell-Masser for instance imply that there are no non trivial polynomial solutions if $1/a + 1/b + 1/c < 1/3$. This condition in particular requires $\min(a, b, c) \geq 4$. Our methods also cover the cases when the minimal exponent is 2 or 3, supposing the remaining two are large enough. The conclusion is that the solutions fall within certain explicitly described families, of which the typical example comes from an identity $(1 + x^a)^2 = 1 + 2x^a + x^{2a}$. This will be done in Theorem 1.2. We remark that the method also works for rational functions x, y, z on higher genus curves; in that case one can obtain bounds for the degrees of the solutions in terms of the genus.

This application can be viewed as a non-existence or finiteness statement for rational curves on a fixed surface. In Theorem 1.2 bis we also consider genus one curves with a finiteness statement.

A natural extension concerns the study of sections of a fibration whose fibers are surfaces. For instance we can consider the case of a fibration of the form $X \to \mathscr{C}$, for a threefold X and a curve $\mathscr{C}$, where the generic fiber is a Fermat-like surface. This amounts for instance to an equation of the form $f(t)x^a + g(t)y^b + h(t) = z^c$, for coefficients $f(t), g(t), h(t) \in \kappa(t)$. We can obtain a bound for the degree of the solutions; again, we do not state it explicitly, and leave it to the interested reader.

Going back to (1.1), we shall assume that

$$z \neq 0, 1, u, v \tag{1.2}$$

and let S_z be the minimal set containing S such that z is an S_z-unit: it is the union

$$S_z := S \cup z^{-1}(0).$$

We also put

$$\tilde{H} := H(1 : u : v) = H(1 : u : v : z), \qquad H^* := \tilde{H} + \chi + \sharp S.$$

Since we are assuming $\sharp(S) \geq 2$, we have

$$H^* > \tilde{H} = H(1 : u : v : z) \geq \max\{H(u), H(v), H(z)\}.$$

Our first result is a lower bound for the number of zeros of z; this is sharp when χ is fixed, or small compared to $\tilde{H}$:

THEOREM 1.1. — *Let u, v be S-units, not both constant, let $z = u+v+1$ satisfy (1.2). Then if u, v are multiplicatively independent modulo κ^* the number of zeros of z outside S satisfies the lower bound*

$$\sharp(S_z \setminus S) = \sum_{\nu \notin S;\, \nu(z)>0} 1 \geq \tilde{H} - 15\chi - 6 \cdot H^{*2/3}\chi^{1/3}.$$

If instead u, v are multiplicatively dependent modulo constants, there is a relation $u^r = \lambda v^s$ with $\lambda \in \kappa^*$, r, s coprime integers, and

$$\sum_{\nu \notin S;\, \nu(z)>0} 1 \geq \tilde{H}\left(1 - \frac{1}{\max(|r|, |s|)}\right) - 15\chi.$$

REMARK. — Since $\tilde{H} \geq \deg(z)$, the above estimates can also be viewed as upper bounds for the number of multiple zeros of z. Note that when z is a square, the left-hand side above is $\leq H(z)/2$. In this case we obtain a bound for its height in term only of χ, unless u, v are multiplicatively dependent modulo κ^*, satisfying a relation with exponents ≤ 2. In that case, however, z can be a square as follows from the identity: $z = (w+w^{-1}/2)^2 = w^2+w^{-2}/4+1$. We also note that qualitative estimates for the number of multiple zeros of polynomial expressions $P(u, v)$ for S-units u, v appear in [C-Z, Theorem 1.3]. The above result provides a completely explicit estimate in the case $P(u, v) = 1 + u + v$; this may also be seen as a special case of an *abcd* theorem.

It will be convenient also to give an alternative statement, of the first case, in terms of an upper bound for H^*: