Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

THE 4-STRING BRAID GROUP B_4 HAS PROPERTY RD AND EXPONENTIAL MESOSCOPIC RANK

Sylvain Barré & Mikaël Pichot

Tome 139
Fascicule 4

2011

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 479-502

Bull. Soc. math. France
139 (4), 2011, p. 479–502

THE 4-STRING BRAID GROUP B_4 HAS PROPERTY RD AND EXPONENTIAL MESOSCOPIC RANK

by Sylvain Barré & Mikaël Pichot

Abstract. — We prove that the braid group B_4 on 4 strings, its central quotient $B_4/\langle z\rangle$, and the automorphism group $\mathrm{Aut}(F_2)$ of the free group F_2 on 2 generators, have the property RD of Haagerup–Jolissaint.

We also prove that the braid group B_4 is a group of intermediate mesoscopic rank (of dimension 3). More precisely, we show that the above three groups have exponential mesoscopic rank, i.e., that they contain exponentially many large flat balls which are not included in flats.

Résumé (*Le groupe de tresses B_4 est de rang mésoscopique et a la propriété RD*)

Nous montrons que le groupe de tresses à 4 brins B_4, son quotient central $B_4/\langle z\rangle$, ainsi que le groupe d'automorphismes $\mathrm{Aut}(F_2)$ du groupe libre à 2 générateurs, possèdent la propreté RD de décroissance rapide de Haagerup–Jolissaint.

Nous montrons également que le groupe de tresses B_4 est un groupe (de dimension 3) de rang intermédiaire mésoscopique. Plus précisément, nous montrons que les trois groupes précédents sont de rang mésoscopique exponentiel, c'est-à-dire qu'ils contiennent un nombre exponentiel de boules plates qui ne sont pas contenues dans des plats.

Texte reçu le 3 février 2009, révisé le 29 octobre 2009, accepté le 13 novembre 2009.

Sylvain Barré, Sylvain Barré, Université de Bretagne Sud, BP 573, Centre Yves Coppens, Campus de Tohannic, 56017 Vannes, France • *E-mail :* `sylvain.barre@univ.ubs.fr`
Mikaël Pichot, Mikaël Pichot, Department of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo, 153-8914 Japan • *E-mail :* `pichot@ms.u-tokyo.ac.jp`

2000 Mathematics Subject Classification. — 20F65.

Key words and phrases. — Braid groups, property RD, CAT(0) spaces.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2011/479/$ 5.00

1. Introduction

Let $n \geq 2$ be an integer. The braid group B_n on n strings is a finitely presented group generated by $n-1$ elementary braids $\sigma_1, \ldots, \sigma_{n-1}$ subject to the following relations:

- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all $1 \leq i \leq n-2$;
- $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $1 \leq i, j \leq n-1$ such that $|i-j| \geq 2$.

This is the classical Artin presentation of B_n (see e.g. Chapter 10 in [11]).

The group B_3 is closely related to the modular group $\mathrm{PSL}_2(\mathbf{Z})$. The above presentation shows that the braid $z = (\sigma_1\sigma_2)^3$ is central in B_3 and that $B_3/\langle z\rangle$ is generated by the class u of $\sigma_1\sigma_2\sigma_1$ and v of $\sigma_1\sigma_2$, where $u^2 = v^3 = z$. Thus $B_3/\langle z\rangle = \langle u, v \mid u^2 = v^3 = 1\rangle = \mathrm{PSL}_2(\mathbf{Z})$. In fact the group B_3 admits a proper isometric action with compact quotient on a metric product $T_3 \times \mathbf{R}$, where T_3 is a trivalent tree, which is the Bass-Serre tree of $\mathrm{PSL}_2(\mathbf{Z})$.

We are interested here in the 4-string braid groups B_4. It was proved by Brady in [7] that B_4 admits a free isometric action with compact quotient on a CAT(0) simplicial complex Y of dimension 3. The 3-dimensional cells of Y are Euclidean tetrahedra whose faces are right-angle triangles and the quotient space Y/B_4 contains 16 tetrahedra, identified together along a single vertex. It is still true that Y splits as a product $Y = X \times \mathbf{R}$, where X is now of dimension 2. The complex X can be obtained from a non positively curved complex of groups whose fundamental group is the quotient of B_4 by its center (see [19]).

The existence of a CAT(0) structure on B_n is an open problem for $n \geq 6$. Recall that on B_4, the 3-dimensional CAT(0) structure which are minimal (e.g., those whose links are isomorphic to that of Y) can be classified, by geometric rigidity results due to Crisp and Paoluzzi [19]. On the other hand, Charney [14] proved that the Deligne complex [20] of B_4 is also a CAT(0) space of dimension 3, with respect to the Moussong metric (we remind that the Deligne action of B_4 on this complex is not proper).

1.1. Property RD.

— Let now G be an arbitrary countable group. A length on G is a map $|\cdot| : G \to \mathbf{R}_+$ such that $|e| = 0$, $|s| = |s^{-1}|$ and $|st| \leq |s| + |t|$ for $s, t \in G$ and e the identity element. We recall that G is said to have *property RD* ([27]) with respect to a length $|\cdot|$ if there is a polynomial P such that for any $r \in \mathbf{R}_+$ and $f, g \in \mathbf{C}G$ with $\mathrm{supp}(f) \subset B_r$ one has

$$\|f * g\|_2 \leq P(r)\|f\|_2\|g\|_2$$

where $B_r = \{x \in G,\ |x| \leq r\}$ is the ball of radius r in G, $\mathrm{supp}(f)$ is the set of $x \in G$ with $f(x) \neq 0$, and $\mathbf{C}G$ is the complex group algebra of G. For an introduction to property RD we refer to Chapter 8 in [37]. The above convolution inequality, usually referred to as the Haagerup inequality (after

Haagerup [25]), allows to control the operator norm of f acting by convolution on $\ell^2(G)$ in terms of its ℓ^2 norm. Hence, some important consequences of property RD are of a spectral nature.

When G is finitely generated we have the word length $|\cdot|_S$ associated to any finite generating set S. Then property RD with respect to $|\cdot|_S$ is independent of S so we simply speak of property RD for G is that case.

Our first main result is the following theorem.

THEOREM 1. — The following groups have the property RD of Haagerup–Jolissaint:

1. the braid group B_4 on 4 strings;
2. the central quotient $B_4/\langle z\rangle$;
3. the automorphism group $\mathrm{Aut}(F_2)$ of the free group on 2 generators.

Theorem 1 shows property RD for low indices in two series of groups, namely:

- the braid groups B_n for $n \geq 3$;
- the automorphism groups $\mathrm{Aut}(F_n)$ for $n \geq 2$.

Property RD for these series is an open problem formulated in Section 8 of [18]. The fact that B_3 has property RD was shown very early on by Jolissaint in [27] and the other cases were left open since then. Shortly after the first version of the present paper appeared, the question of showing property RD for all groups B_n has been answered by Behrstock and Minsky (see [6]). More generally, they established property RD for all mapping class groups. (Recall that the braid group B_n can be identified to the mapping class group of the n-punctured disk.) The problem for $\mathrm{Aut}(F_n)$, $n \geq 3$, remains open.

The proof of Theorem 1 is divided into two steps. The first step relies on our previous results from [4]:

THEOREM 2 ([4, Theorem 5]). — Let G be a group acting properly on a CAT(0) simplicial complex Δ of dimension 2 without boundary and whose faces are equilateral triangles of the Euclidean plane. Then G has property RD with respect to the length induced from the 1-skeleton of Δ.

We apply Theorem 2 to the quotient $B_4/\langle z\rangle$. By results of [7, 19], this group acts on a simplicial complex X with the required properties.

The second step uses automaticity of B_4, and more precisely, the Thurston normal forms for braids in B_4, which allows to go back to B_4 from its central quotient.

We now show how deduce (3) from (2) in Theorem 1. The group $\mathrm{Aut}(F_2)$ is isomorphic to $\mathrm{Aut}(B_4)$, itself containing $\mathrm{Inn}(B_4)$ as a subgroup of index 2 (see [21, 28]). Thus property RD for $\mathrm{Aut}(F_2)$ follows from the corresponding result for $\mathrm{Inn}(B_4)$. Then (3) follows from the fact that $\mathrm{Inn}(B_4)$ is isomorphic to the central quotient of B_4. Note that in [32], a faithful action of $\mathrm{Aut}(F_2) = \mathrm{Aut}(B_4)$ on the complex X is constructed.

Details of the proof are in Section 3, after a brief survey on property RD in Section 2.

1.2. The braid group B_4 as a group of intermediate rank. — Groups and simplicial complexes appearing in Theorem 2 provide us with a large pool of objects satisfying *intermediate rank* properties. See [4] for definitions and concrete examples. We discuss here the intermediate rank properties of B_4 and its central quotient (denoted G below).

We introduced in [4] a notion of *mesoscopic rank* for a CAT(0) space X, which reflects the presence in X of maximal flats portions (where maximal refers to the dimension, hence the rank terminology) which are (much) larger than "germs of flats" in X (say, flats of tangent cones) but *are not actually contained in genuine flats of* X (i.e. copies of the Euclidean space $\mathbf{R}^n$ inside X). We recall the precise definitions of mesoscopic rank and exponential mesoscopic rank in Section 5. Following [4] we say that a group G is of (exponential) mesoscopic rank when there is a proper action of G with compact quotient on some CAT(0) space which is of (exponential) mesoscopic rank at some point.

Our second main result is as follows.

THEOREM 3. — The braid group B_4 on 4 strings is of exponential mesoscopic rank.

For the proof, we first establish that the quotient G of B_4 by its center is of exponential mesoscopic rank, and then extend the result to B_4. Note that B_3 is an example of a group acting freely and cocompactly on a simplicial complex as in Theorem 2 (see [8]) but it is not of mesoscopic rank, and more precisely for any action with compact quotient on a 2-dimensional CAT(0) space X, the space X cannot be of mesoscopic rank.

In course of proving Theorem 3 we will see that the central quotient G of B_4 is, at the local level, closely related to affine Bruhat-Tits buildings of type $\tilde{A}_2$ (what actually creates some complications in the proof of Theorem 3, since the latter are not of mesoscopic rank by [4]). We will prove however that these connections cannot be extended beyond the local level (and specifically beyond the sphere of radius 1, see the last section of the paper).

Acknowledgments. — We thank Jason Behrstock for communicating us his recent preprint [6] with Yair Minsky as well as the reference [32]. The second author thanks JSPS for support.