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David Chataur & Jean-François Le Borgne

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ON THE LOOP HOMOLOGY OF COMPLEX PROJECTIVE SPACES

BY DAVID CHATAUR & JEAN-FRANÇOIS LE BORGNE

ABSTRACT. — In this short note we compute the Chas-Sullivan BV-algebra structure on the singular homology of the free loop space of complex projective spaces. We compare this result with computations in Hochschild cohomology.

RÉSUMÉ (*Sur l'homologie des lacets d'espaces projectifs complexes*)

Sur l'homologie de l'espace des lacets des espaces projectifs complexes résumé : Dans cette note, on calcule l'homologie singulière de l'espace des lacets libres des espaces projectifs complexes munie de la BV-structure de Chas-Sullivan. On compare ces calculs avec ceux effectués en cohomologie de Hochschild.

Introduction

Let us begin by recalling the definitions of *Gerstenhaber* algebras and *BV-algebras*. A Gerstenhaber algebra is a triple $(A, \bullet, \{-, -\})$ such that (A, \bullet) is a

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DAVID CHATAUR, Laboratoire Paul Painlevé, Université de Lille 1, 59655 Villeneuve d'Ascq Cedex, France • *E-mail* : David.Chataur@math.univ-lille1.fr

JEAN-FRANÇOIS LE BORGNE, Laboratoire Paul Painlevé, Université de Lille 1, 59655 Villeneuve d'Ascq Cedex, France • *E-mail* : Leborgne.Jean-Francois@math.univ-lille1.fr

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commutative graded algebra and $(A, \{-, -\})$ is a graded Lie algebra of degree $+ 1$. Moreover the product and the Lie bracket satisfy the following relation the so-called Poisson relation:

$$\{a, b \bullet c\} = \{a, b\} \bullet c + (-1)^{(|a|+1)|b|} b \bullet \{a, c\}.$$

A Batalin-Vilkovisky algebra or BV-algebra is a Gerstenhaber algebra $(A, \bullet, \{-, -\})$ equipped with a degree $+1$ linear map

$$\Delta : A_i \rightarrow A_{i+1}$$

such that $\Delta^2 = 0$ and such that we have the formula

$$\{a, b\} = (-1)^{|a|}(\Delta(a \bullet b) - \Delta(a) \bullet b - (-1)^{|a|} a \bullet \Delta(b)),$$

these algebras are closely related to topological field theories [9].

The first examples of Gerstenhaber algebras go back to Gerstenhaber himself. Let A be an associative algebra, Gerstenhaber proved that the Hochschild cohomology of A denoted by $HH^*(A, A)$ is a Gerstenhaber algebra [8]. L. Menichi [13] and T. Tradler [18] proved that for any symmetric algebra A the Hochschild cohomology $HH^*(A, A)$ is a Batalin-Vilkovisky algebra. In this case the Δ -operator is induced by the Connes coboundary map. This result has been reproved and extended by many people. In particular, L. Menichi proved in [14] that the Hochschild cohomology $HH^*(A, A)$ of a differential graded algebra A which is a symmetric algebra at the level of the derived category of algebras is a Batalin-Vilkovisky algebra. Let M be a d -dimensional connected closed oriented manifold and $C^*(M, \mathbb{Z})$ its singular cochain complex, in [7] the authors proved a linear isomorphism

$$D : HH^{*+d}(C^*(M, \mathbb{Z}), C_*(M, \mathbb{Z})) \rightarrow HH^*(C^*(M, \mathbb{Z}), C^*(M, \mathbb{Z})).$$

This linear isomorphism depends on choices of $C^*(M, \mathbb{Z})$ -bimodules quasi-isomorphisms between chains and cochains of M . And it is required that these quasi-isomorphisms induce the Poincaré duality at the homology level. L. Menichi’s results imply that the Connes coboundary map on Hochschild cohomology $HH^*(C^*(M, \mathbb{Z}), C_*(M, \mathbb{Z}))$ defines via the isomorphism D a structure of BV algebra extending the Gerstenhaber algebra $HH^*(C^*(M, \mathbb{Z}), C^*(M, \mathbb{Z}))$.

The second examples of BV-algebras we encounter in this note come from string topology. The free loop space of M denoted by $\mathcal{L}M$ is the space of continuous maps from the circle S^1 to M . In [3], M. Chas and D. Sullivan defined on the shifted homology $\mathbb{H}_*(\mathcal{L}M) := H_{*+d}(\mathcal{L}M, \mathbb{Z})$ a structure of BV algebra. The commutative product is the *loop product* denoted by \circ . This loop product is a kind of intersection product for free loop spaces of manifolds. It mixes the intersection product of the singular homology of M together with the Pontryagin product of the homology of the based loop space $\Omega_m M$. The Δ -operator comes from the natural action of the circle on $\mathcal{L}M$ given by reparametrization

of loops. More precisely, this operator is the composition of the two following maps. First, we consider the morphism

$$\begin{aligned} [S^1] \times : H_*(\mathcal{L}M) &\rightarrow H_{*+1}(S^1 \times \mathcal{L}M) \\ x &\mapsto [S^1] \times x \end{aligned}$$

given by the cross product with the fundamental class of S^1 . Next, we consider

$$act_* : H_{*+1}(S^1 \times \mathcal{L}M) \rightarrow H_{*+1}(\mathcal{L}M)$$

the map induced in homology by the reparametrization action of the circle:

$$\begin{aligned} act : S^1 \times \mathcal{L}M &\rightarrow \mathcal{L}M \\ (\theta, \gamma(t)) &\mapsto \gamma(\theta + t). \end{aligned}$$

For more details on the constructions of the loop product and of the string topology BV-structure we refer to Chas-Sullivan and Cohen-Jones' papers ([3] and [4]).

A fundamental problem in string topology is to compare $(\mathbb{H}(\mathcal{L}M), \circ, \Delta)$ together with $HH^*(C^*(M, \mathbb{Z}), C^*(M, \mathbb{Z}))$. Working over a field \mathbb{F} , in [4] Cohen and Jones proved an isomorphism of graded commutative and associative algebras

$$\mathbb{H}_*(\mathcal{L}M, \mathbb{F}) \cong HH^*(C^*(M, \mathbb{F}), C^*(M, \mathbb{F})).$$

Now, let us suppose that M is a formal manifold, that is to say that $C^*(M, \mathbb{Z})$ and $H^*(M, \mathbb{Z})$ are isomorphic in the homotopy category of differential graded algebras. For example, spheres and projective spaces are formal manifolds. In that cases, in first approximation one can use the structure of Poincaré duality algebra of $H^*(M, \mathbb{Z})$ and try to compare $(\mathbb{H}_*(\mathcal{L}M), \circ, \Delta)$ together with $HH^*(H^*(M, \mathbb{Z}), H^*(M, \mathbb{Z}))$ as BV-algebras. For the manifold $M = \mathbb{C}P^1 = S^2$, L. Menichi in [15] proved that $(\mathbb{H}_*(\mathcal{L}S^2, \mathbb{Z}/2\mathbb{Z}), \circ, \Delta)$ cannot be isomorphic to $HH^*(H^*(S^2, \mathbb{Z}/2\mathbb{Z}), H^*(S^2, \mathbb{Z}/2\mathbb{Z}))$ as a BV-algebra. Then L. Menichi asked for the case of complex projective spaces. It is the aim of this paper to give these computations. We prove the existence of an isomorphism of BV-algebras for $M = \mathbb{C}P^{2m}$ but never for $M = \mathbb{C}P^{2m+1}$, however we always have an isomorphism of Gerstenhaber algebras. All our computations are with homology with integral coefficients, we notice that few computations of loop homology BV-algebras were done over the integers, H. Tamanoi in [17] did it for complex Stiefel manifolds. These computations prove that in string topology the comparison of the Chas-Sullivan BV-algebra with Hochschild cohomology of cochains is very subtle. In fact the main point is to produce a correct notion of homotopy Poincaré duality algebra. And maybe for such a notion complex projective spaces of odd dimension are not formal manifolds. Recently R. Hepworth in [11] has also computed the Chas-Sullivan BV-algebra of complex projective spaces by using different methods.

Statements and results

Chas-Sullivan BV structure. — In [6], the authors proved that rationally for 1-connected closed oriented manifolds, there is an isomorphism of BV algebra between the Hochschild cohomology $HH_*(C^*(M, \mathbb{Q}), C^*(M, \mathbb{Q}))$ and $(\mathbb{H}_*(\mathcal{L}M, \mathbb{Q}), \circ, \Delta)$. As a consequence of this isomorphism T. Yang deduced that ([19, theorem 4.3])

As BV algebras,

$$\mathbb{H}_*(\mathcal{L}CP^n; \mathbb{Q}) = \mathbb{Q}[\alpha_{-2}, \xi_{-1}, \zeta_{2n}] / (\alpha_{-2}^{n+1}, \xi_{-1}^2, \alpha_{-2}^n \xi_{-1}, \zeta_{2n} \alpha_{-2}^n)$$

where the subscripts are the degrees of the generator.

$$\Delta(\zeta_{2n}^k \alpha_{-2}^l) = 0$$

$$\Delta(\zeta_{2n}^k \xi_{-1} \alpha_{-2}^l) = -(k+1)n - k + l \zeta_{2n}^k \alpha_{-2}^l$$

In [5], the authors compute the loop algebra $\mathbb{H}_*(\mathcal{L}CP^n, \mathbb{Z})$. Their computation uses the Leray-Serre spectral sequence of the evaluation fibration

$$\Omega_m \mathbb{C}P^n \rightarrow \mathcal{L}CP^n \xrightarrow{ev_0} \mathbb{C}P^n.$$

They used the fact that this fibration is multiplicative with respect to the loop product and showed that

The algebra $\mathbb{H}_(\mathcal{L}CP^n)$ is isomorphic to*

$$\mathbb{Z}[a_{-2}, x_{-1}, y_{2n}] / (a_{-2}^{n+1}, x_{-1}^2, a_{-2}^n x_{-1}, (n+1)y_{2n} a_{-2}^n)$$

where the subscripts are the degrees of the generators.

One can already notice that Cohen-Jones-Yan’s computations rely upon W. Ziller’s results [20] that are based on Morse theoretic arguments.

Using Morse theory we prove that:

THEOREM 0.1. — *The BV structure on*

$$\mathbb{H}_*(\mathcal{L}CP^n) \simeq \mathbb{Z}[a_{-2}, x_{-1}, y_{2n}] / (a_{-2}^{n+1}, x_{-1}^2, a_{-2}^n x_{-1}, (n+1)y_{2n} a_{-2}^n)$$

is given by

$$\Delta(y_{2n}^k a_{-2}^l) = 0.$$

For $l \neq 0$,

$$\Delta(y_{2n}^k x_{-1} a_{-2}^l) = -(k+1)n - k + l y_{2n}^k a_{-2}^l$$

If n is odd,

$$\Delta(y_{2n}^k x_{-1}) = -(k+1)n - k y_{2n}^k + (k+1) \cdot \frac{n+1}{2} a_{-2}^n y_{2n}^{k+1}.$$

If n is even,

$$\Delta(y_{2n}^k x_{-1}) = -(k+1)n - k + l y_{2n}^k.$$