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# COLEFF-HERRERA CURRENTS, DUALITY, AND NOETHERIAN OPERATORS

BY MATS ANDERSSON

ABSTRACT. — Let $\mathscr{J}$ be a coherent subsheaf of a locally free sheaf $\mathscr{O}(E_0)$ and suppose that $\mathscr{F} = \mathscr{O}(E_0)/\mathscr{J}$ has pure codimension. Starting with a residue current $R$ obtained from a locally free resolution of $\mathscr{F}$ we construct a vector-valued Coleff-Herrera current $\mu$ with support on the variety associated to $\mathscr{F}$ such that $\phi$ is in $\mathscr{J}$ if and only if $\mu\phi = 0$. Such a current $\mu$ can also be derived algebraically from a fundamental theorem of Roos about the bidualizing functor, and the relation between these two approaches is discussed. By a construction due to Björk one gets Noetherian operators for $\mathscr{J}$ from the current $\mu$. The current $R$ also provides an explicit realization of the Dickenstein-Sessa decomposition and other related canonical isomorphisms.

RÉSUMÉ (*Courants de Coleff-Herrera, dualité et opérateurs noethériens*)
Soit $\mathscr{J}$ un sous-faisceau cohérent d'un faisceau localement libre $\mathscr{O}(E_0)$ et supposons que $\mathscr{F} = \mathscr{O}(E_0)/\mathscr{J}$ ait une codimension pure. En partant d'un courant résiduel $R$, obtenu à partir d'une résolution localement libre de $\mathscr{F}$, nous construisons un courant de Coleff-Herrera vectoriel $\mu$ à support sur la variété associée à $\mathscr{F}$, tel que $\phi$ soit dans $\mathscr{J}$ si et seulement si $\mu\phi = 0$. Un tel courant $\mu$ peut également être dérivé algébriquement grâce à un théorème fondamental de Roos sur le foncteur bidualisant, et nous étudions le lien entre les deux approches. Par une construction due à Björk, on obtient des opérateurs noethériens pour $\mathscr{J}$ à partir du courant $\mu$. Le courant $R$ nous fournit également une réalisation explicite de la décomposition de Dickenstein-Sessa, ainsi que d'autres isomorphismes canoniques afférents.

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## 1. Introduction

A function $\phi$ in the local ring $\mathscr{O}_0$ in one complex variable belongs to the ideal $I$ generated by $z^m$ if and only if

$$\mathscr{L}_\ell \phi(0) = 0, \ \ell = 0, \dots, m-1,$$

where $\mathscr{L}_\ell = \partial^\ell / \partial z^\ell$. These conditions can be elegantly expressed by the single equation $\phi \bar\partial(1/z^m) = 0$, where $1/z^m$ is the usual principal value distribution. Moreover, the current $\mu = \bar\partial(1/z^m)$ is canonical up to a non-vanishing holomorphic factor. There is a well-known multivariable generalization. Let $f = (f^1, \dots, f^p)$ be a tuple of holomorphic functions in a neighborhood of the origin in $\mathbb{C}^n$ that defines a complete intersection, i.e., the codimension of $Z^f = \{f = 0\}$ is equal to $p$. Then the Coleff-Herrera product

$$\mu^f = \bar\partial \frac{1}{f^1} \wedge \cdots \wedge \bar\partial \frac{1}{f^p},$$

introduced in [9], is a $\bar\partial$-closed $(0,p)$-current with support on $Z^f$, and it is independent (up to a nonvanishing holomorphic factor) of the choice of generators of the ideal sheaf $\mathscr{I}$ generated by $f$. It was proved in [10] and [17] that $\mathscr{I}$ coincides with the ideal sheaf $\operatorname{ann} \mu^f$ of holomorphic functions $\phi$ such that the current $\mu^f \phi$ vanishes. This is often referred to as the duality principle.

The Coleff-Herrera product is the model for a general Coleff-Herrera current introduced by Björk: Given a variety $Z$ of pure codimension $p$ we say that a (possibly vector-valued) $(0,p)$-current $\mu$ (with support on $Z$) is a Coleff-Herrera current on $Z$, $\mu \in \mathscr{CH}_Z$, if it is $\bar\partial$-closed, annihilated by $\bar{\mathscr{I}}_Z$ (i.e., $\bar\xi \mu = 0$ for each holomorphic $\xi$ that vanishes on $Z$), and has the standard extension property SEP. This means, roughly speaking, that $\mu$ has no "mass" concentrated on any subvariety of higher codimension; in particular that $\mu$ is determined by its values on $Z_{reg}$, see, e.g., [7] or [3], and Section 2.1. The SEP implies that $\operatorname{ann} \mu$ has pure dimension, see, e.g., Proposition 5.3 in [5]. The condition $\bar{\mathscr{I}}_Z \mu = 0$ means that $\mu$ only involves holomorphic derivatives. Following Björk, see [7], one can quite easily find a finite number of holomorphic differential operators $\mathscr{L}_\ell$ such that $\phi\mu = 0$ if and only if $\mathscr{L}_1 \phi = \cdots = \mathscr{L}_\nu \phi = 0$ on $Z$; i.e., a (complete) set of Noetherian operators for $\operatorname{ann} \mu$. In this paper we use the residue theory developed in [4] and [5] to extend the duality for a complete intersection to a general pure-dimensional ideal (or submodule of a locally free) sheaf. In particular we can express such an ideal as the annihilator of a finite set of Coleff-Herrera currents (Theorem 1.2 and its corollaries). Jan-Erik Björk has pointed out to us that one can deduce the same duality result from a fundamental theorem of Jan-Erik Roos, [18], about purity for a module in terms of the bidualizing sheaves, combined with some other known facts that will be described below. However our approach gives a representation of the duality and the Coleff-Herrera currents in terms of one basic residue current, that we first describe.

To begin with, let $\mathcal{J}$ be any coherent subsheaf of a locally free sheaf $\mathcal{O}(E_0)$ over a complex manifold $X$, and assume that

$$0 \to \mathcal{O}(E_N) \xrightarrow{f_N} \cdots \xrightarrow{f_3} \mathcal{O}(E_2) \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0) \tag{1.1}$$

is a locally free resolution of $\mathcal{F} = \mathcal{O}(E_0)/\mathcal{J}$. Here $\mathcal{O}(E_k)$ denotes the locally free sheaf associated to the vector bundle $E_k$ over $X$. If $X$ is Stein, then one can find such a resolution in a neighborhood of any given compact subset. We will assume that $\mathcal{F}$ has codimension $p > 0$; cf., Remark 2. Then $f_1$ is (can be assumed to be) generically surjective, and the analytic set $Z$ where it is not surjective has codimension $p$ and coincides with the zero set of the ideal sheaf $\operatorname{ann}\mathcal{F}$. In [4] we defined, given Hermitian metrics on $E_k$, a residue current $R = R_p + R_{p+1} + \cdots$ with support on $Z$, where $R_k$ is a $(0,k)$-current that takes values in $\operatorname{Hom}(E_0, E_k)$, such that a holomorphic section $\phi \in \mathcal{O}(E_0)$ is in $\mathcal{J}$ if and only if $R\phi = 0$.

Recall that $\mathcal{F}$ has *pure* codimension $p$ if the associated prime ideals (of each stalk) all have codimension $p$. The starting point in this paper is the following result that follows from [5] (see also Section 7 below); as we will see later on it is in a way equivalent to Roos' characterization of purity.

THEOREM 1.1. — *The sheaf $\mathcal{F} = \mathcal{O}(E_0)/\mathcal{J}$ has pure codimension $p$ if and only if $\mathcal{J}$ is equal to the annihilator of $R_p$, i.e.,*

$$\mathcal{J} = \{\phi \in \mathcal{O}(E_0);\ R_p\phi = 0\}.$$

If $\mathcal{F}$ is Cohen-Macaulay we can choose a resolution (1.1) with $N = p$, and then $R = R_p$ is a matrix of $\mathcal{CH}_Z$-currents which thus solves our problem. However, in general $R_p$ is not $\bar\partial$-closed even if $\mathcal{F}$ has pure codimension. Let

$$0 \to \mathcal{O}(E_0^*) \xrightarrow{f_1^*} \mathcal{O}(E_1^*) \xrightarrow{f_2^*} \cdots \xrightarrow{f_{p-1}^*} \mathcal{O}(E_{p-1}^*) \xrightarrow{f_p^*} \mathcal{O}(E_p^*) \xrightarrow{f_{p+1}^*} \tag{1.2}$$

be the dual complex of (1.1) and let

$$\mathcal{H}^k(\mathcal{O}(E_\bullet^*)) = \frac{\operatorname{Ker}_{f_{k+1}^*} \mathcal{O}(E_k^*)}{f_k^*\mathcal{O}(E_{k-1}^*)} \tag{1.3}$$

be the associated cohomology sheaves. It turns out that for each choice of $\xi \in \mathcal{O}(E_p^*)$ such that $f_{p+1}^*\xi = 0$, the current $\xi R_p$ is in $\mathcal{CH}_Z(E_0^*)$, and we have in fact a bilinear (over $\mathcal{O}$) pairing

$$\mathcal{H}^p(\mathcal{O}(E_\bullet^*)) \times \mathcal{F} \to \mathcal{CH}_Z, \quad (\xi, \phi) \mapsto \xi R_p \phi. \tag{1.4}$$

Moreover, (1.4) is independent of the choice of Hermitian metrics on $E_k$. It is well-known that the sheaves in (1.3) represent the intrinsic sheaves $\mathcal{E}xt^k_{\mathcal{O}}(\mathcal{F}, \mathcal{O})$. (If $Z$ does not have pure codimension $p$ then we define $\mathcal{CH}_Z$ as $\mathcal{CH}_{Z'}$, where

$Z'$ is the union of irreducible components of codimension $p$; this is reasonable, in view of the SEP.)

THEOREM 1.2. — *Assume that $\mathcal{F}$ has codimension $p > 0$. The pairing (1.4) induces an intrinsic pairing*

$$\mathcal{E}xt^p_{\mathcal{O}}(\mathcal{F}, \mathcal{O}) \times \mathcal{F} \to \mathcal{CH}_Z. \tag{1.5}$$

*If $\mathcal{F}$ has pure codimension, then the pairing is non-degenerate.*

Notice that $\mathcal{H}om\,(\mathcal{F}, \mathcal{CH}_Z)$ is the subsheaf of $\mathcal{H}om\,(\mathcal{O}(E_0), \mathcal{CH}_Z) = \mathcal{CH}_Z(E_0^*)$ consisting of all Coleff-Herrera currents $\mu$ with values in $E_0^*$ such that $\mu\phi = 0$ for all $\phi \in \mathcal{J}$. It follows that we have the equality

$$\mathcal{J} = \{\phi \in \mathcal{O}(E_0);\ \mu\phi = 0 \text{ for all } \mu \in \mathcal{H}om\,(\mathcal{F}, \mathcal{CH}_Z)\} \tag{1.6}$$

if $\mathcal{F}$ is pure. The sheaf $\mathcal{H}^p(\mathcal{O}(E_\bullet^*))$ is coherent and thus locally finitely generated. Therefore we have now a solution to our problem:

COROLLARY 1.3. — *Assume that $\mathcal{F}$ has pure codimension. If $\xi_1, \ldots, \xi_\nu \in \mathcal{O}(E_p^*)$ generate $\mathcal{H}^p(\mathcal{O}(E_\bullet^*))$, then $\mu_j = \xi_j R_p$ are in $\mathcal{H}om\,(\mathcal{F}, \mathcal{CH}_Z)$ and*

$$\mathcal{J} = \cap_{j=1}^{\nu} \operatorname{ann} \mu_j. \tag{1.7}$$

REMARK 1. — If $\mathcal{J}$ is not pure, one obtains a decomposition (1.7) after a preliminary decomposition $\mathcal{J} = \cap \mathcal{J}_\nu$, where each $\mathcal{J}_\nu$ has pure codimension. □

In case of a complete intersection, $\mathcal{E}xt^p(\mathcal{F}, \mathcal{O})$ is isomorphic to $\mathcal{F}$ itself. If $\mathcal{F} = \mathcal{O}(E_0)/\mathcal{J}$ is a sheaf of Cohen-Macaulay modules there is also a certain symmetry: If (1.1) is a resolution with $N = p$, then it is well-known, cf., also Example 4 below, that the dual complex (1.2) is a resolution of $\mathcal{O}(E_p^*)/\mathcal{J}^*$, where $\mathcal{J}^* = f_p^* \mathcal{O}(E_{p-1}^*) \subset \mathcal{O}(E_p^*)$, and we have

COROLLARY 1.4. — *If $\mathcal{O}(E_0)/\mathcal{J}$ is Cohen-Macaulay, then $\mathcal{O}(E_p^*)/\mathcal{J}^*$ is Cohen-Macaulay as well and we have a non-degenerate pairing*

$$\mathcal{O}(E_0)/\mathcal{J} \times \mathcal{O}(E_p^*)/\mathcal{J}^* \to \mathcal{CH}_Z, \quad (\xi, \phi) \mapsto \xi R_p \phi.$$

REMARK 2. — Assume that $\mathcal{F}$ has codimension $p = 0$, or equivalently, $\operatorname{ann} \mathcal{F} = 0$. If it is pure, i.e., (0) is the only associated prime ideal, then there is a homomorphism $f_0 \colon \mathcal{O}(E_0) \to \mathcal{O}(E_{-1})$ such that $\mathcal{J} = \operatorname{Ker} f_0$. It is natural to consider $f_0$ as a Coleff-Herrera current $\mu$ associated with the zero-codimensional "variety" $X$. Then $\mathcal{J} = \operatorname{ann} \mu$ and thus analogues of Theorem 1.1 and Corollary 1.3 still hold. □