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COMPATIBILITY OF THE THETA CORRESPONDENCE WITH THE WHITTAKER FUNCTORS

BY VINCENT LAFFORGUE & SERGEY LYSENKO

ABSTRACT. — We prove that the global geometric theta-lifting functor for the dual pair (H, G) is compatible with the Whittaker functors, where (H, G) is one of the pairs $(\mathrm{S}\mathbb{O}_{2n}, \mathbb{S}\mathrm{p}_{2n})$, $(\mathbb{S}\mathrm{p}_{2n}, \mathrm{S}\mathbb{O}_{2n+2})$ or $(\mathbb{G}\mathrm{L}_n, \mathbb{G}\mathrm{L}_{n+1})$. That is, the composition of the theta-lifting functor from H to G with the Whittaker functor for G is isomorphic to the Whittaker functor for H.

RÉSUMÉ (*Compatibilité de la thêta-correspondence avec les foncteurs de Whittaker*)
Nous démontrons que le foncteur géométrique de théta-lifting pour la paire duale (H, G) est compatible avec la normalisation de Whittaker, où (H, G) est l'une des paires $(\mathrm{S}\mathbb{O}_{2n}, \mathbb{S}\mathrm{p}_{2n})$, $(\mathbb{S}\mathrm{p}_{2n}, \mathrm{S}\mathbb{O}_{2n+2})$ ou $(\mathbb{G}\mathrm{L}_n, \mathbb{G}\mathrm{L}_{n+1})$. Plus précisément, le composé du foncteur de théta-lifting de H vers G et du foncteur de Whittaker pour G est isomorphe au foncteur de Whittaker pour H.

We prove in this note that the global geometric theta lifting for the pair (H, G) is compatible with the Whittaker normalization, where $(H, G) = (\mathrm{S}\mathbb{O}_{2n}, \mathbb{S}\mathrm{p}_{2n})$, $(\mathbb{S}\mathrm{p}_{2n}, \mathrm{S}\mathbb{O}_{2n+2})$, or $(\mathbb{G}\mathrm{L}_n, \mathbb{G}\mathrm{L}_{n+1})$. More precisely, let k be an algebraically closed field of characteristic $p > 2$. Let X be a smooth projective connected curve over k. For a stack S write $\mathrm{D}(S)$ for the derived category of étale constructible $\bar{\mathbb{Q}}_\ell$-sheaves on S. For a reductive group G over k write

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Bun_G for the stack of G-torsors on X. The usual Whittaker distribution admits a natural geometrization $\mathrm{Whit}_G : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Spec}\, k)$.

We construct an isomorphism of functors between $\mathrm{Whit}_G \circ F$ and Whit_H, where $F : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ is the theta lifting functor (cf. Theorems 1, 2 and 3).

This result at the level of functions (on $\mathrm{Bun}_H(k)$ and $\mathrm{Bun}_G(k)$ when k is a finite field) is well known since a long time and the geometrization of the argument is straightforward. We wrote this note for the following reason.

Our proof holds also for $k = \mathbb{C}$ in the setting of D-modules. In this case for a reductive group G, Beilinson and Drinfeld proposed a conjecture, which (in a form that should be made more precise) says that there exists an equivalence α_G between the derived category of D-modules on Bun_G and the derived category of $\mathscr{O}$-modules on $\mathrm{Loc}_{\check{G}}$. Here $\mathrm{Loc}_{\check{G}}$ is the stack of $\check{G}$-local systems on X, and $\check{G}$ is the Langlands dual group to G. Moreover, Whit_G should be the composition $\mathrm{D}(D\text{-mod}(\mathrm{Bun}_G)) \stackrel{\alpha_G}{\to} \mathrm{D}(\mathrm{Loc}_{\check{G}}, \mathscr{O}) \stackrel{\mathrm{R}\Gamma}{\to} \mathrm{D}(\mathrm{Spec}\, \mathbb{C})$.

A morphism $\gamma : \check{H} \to \check{G}$ gives rise to the extension of scalars morphism $\bar{\gamma} : \mathrm{Loc}_{\check{H}} \to \mathrm{Loc}_{\check{G}}$. The functor $\bar{\gamma}_* : \mathrm{D}(\mathrm{Loc}_{\check{H}}, \mathscr{O}) \to \mathrm{D}(\mathrm{Loc}_{\check{G}}, \mathscr{O})$ should give rise to the Langlands functoriality functor

$$\gamma_L = \alpha_G^{-1} \circ \bar{\gamma}_* \circ \alpha_H : \mathrm{D}(D\text{-mod}(\mathrm{Bun}_H)) \to \mathrm{D}(D\text{-mod}(\mathrm{Bun}_G))$$

compatible with the action of Hecke functors.

In the cases $(H, G) = (\mathbb{SO}_{2n}, \mathbb{S}\mathrm{p}_{2n})$, $(\mathbb{S}\mathrm{p}_{2n}, \mathbb{SO}_{2n+2})$ or $(\mathbb{GL}_n, \mathbb{GL}_{n+1})$ the compatibility of the theta lifting functor $F : \mathrm{D}(D\text{-mod}(\mathrm{Bun}_H)) \to D(D\text{-mod}(\mathrm{Bun}_G))$ with the Hecke functors ([6]) and the compatibility of F with the Whittaker functors (proved in this paper) indicate that F should be the Langlands functoriality functor.

Notation. From now on k denotes an algebraically closed field of characteristic $p > 2$, all the stacks we consider are defined over k. Let X be a smooth projective curve of genus g. Fix a prime $\ell \neq p$ and a non-trivial character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$, and denote by $\mathscr{L}_\psi$ the corresponding Artin-Schreier sheaf on $\mathbb{A}^1$. Since k is algebraically closed, we systematically ignore the Tate twists.

For a k-stack locally of finite type S write simply $\mathrm{D}(S)$ for the category introduced in ([3], Remark 3.21) and denoted $\mathrm{D}_c(S, \bar{\mathbb{Q}}_\ell)$ in *loc.cit.* It should be thought of as the unbounded derived category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves on S. For $* = +, -, b$ we have the full triangulated subcategory $\mathrm{D}^*(S) \subset \mathrm{D}(S)$ denoted $\mathrm{D}_c^*(S, \bar{\mathbb{Q}}_\ell)$ in *loc.cit.* Write $\mathrm{D}^*(S)_! \subset \mathrm{D}^*(S)$ for the full subcategory of objects which are extensions by zero from some open substack of finite type. Write $\mathrm{D}^{\prec}(S) \subset \mathrm{D}(S)$ for the full subcategory of complexes $K \in D(S)$ such that for any open substack $U \subset S$ of finite type we have $K \mid_U \in D^-(U)$.

For any vector space (or bundle) E, we define $\mathrm{Sym}^2(E)$ and $\Lambda^2(E)$ as quotients of $E\otimes E$ (and denote by $x.y$ and $x\wedge y$ the images of $x\otimes y$) and we will use in this article the embeddings

$$\begin{array}{llll} \mathrm{Sym}^2(E) \to & E\otimes E & \text{and} \quad \Lambda^2(E) \to & E\otimes E \\ x.y \quad \mapsto & \frac{x\otimes y + y\otimes x}{2} & \quad x\wedge y \;\mapsto & \frac{x\otimes y - y\otimes x}{2} \end{array} \tag{1}$$

1. Whittaker functors

Let G be a reductive group over k. We pick a maximal torus and a Borel subgroup $T\subset B\subset G$ and we denote by Δ_G the set of simple roots of G. The Whittaker functor

$$\mathrm{Whit}_G : \mathrm{D}^{\prec}(\mathrm{Bun}_G)\to \mathrm{D}^{-}(\mathrm{Spec}\, k)$$

is defined as follows. Write Ω for the canonical line bundle on X. Pick a T-torsor $\mathcal{F}_T$ on X with a trivial conductor, that is, for each $\check\alpha\in\Delta_G$ it is equipped with an isomorphism $\delta_{\check\alpha} : \mathcal{L}^{\check\alpha}_{\mathcal{F}_T}\to\Omega$. Here $\mathcal{L}^{\check\alpha}_{\mathcal{F}_T}$ is the line bundle obtained from $\mathcal{F}_T$ via extension of scalars $T\xrightarrow{\check\alpha}\mathbb{G}_m$. Let $\mathrm{Bun}_N^{\mathcal{F}_T}$ be the stack classifying a B-torsor $\mathcal{F}_B$ together with an isomorphism

$$\zeta : \mathcal{F}_B\times_B T\widetilde{\to}\mathcal{F}_T$$

Let $\epsilon : \mathrm{Bun}_N^{\mathcal{F}_T}\to\mathbb{A}^1$ be the evaluation map (cf. [1], 4.3.1 where it is denoted $ev_{\tilde\omega}$). Just recall that for each $\check\alpha\in\Delta_G$ the class of the extension of $\mathcal{O}$ by Ω associated to $\mathcal{F}_B$, ζ and $\delta_{\check\alpha}$ gives $\epsilon_{\check\alpha} : \mathrm{Bun}_N^{\mathcal{F}_T}\to\mathbb{A}^1$ and that $\epsilon=\sum_{\check\alpha\in\Delta_G}\epsilon_{\check\alpha}$. Write $\pi : \mathrm{Bun}_N^{\mathcal{F}_T}\to\mathrm{Bun}_G$ for the extension of scalars $(\mathcal{F}_B,\zeta)\mapsto\mathcal{F}_B\times_B G$. Set $P^0_\psi=\epsilon^*\mathcal{L}_\psi[d_N]$, where $d_N=\dim\mathrm{Bun}_N^{\mathcal{F}_T}$. Let $d_G=\dim\mathrm{Bun}_G$. As in ([7], Definition 2) for $\mathcal{F}\in\mathrm{D}^{\prec}(\mathrm{Bun}_G)$ set

$$\mathrm{Whit}_G(\mathcal{F})=\mathrm{R}\Gamma_c(\mathrm{Bun}_N^{\mathcal{F}_T}, P^0_\psi\otimes\pi^*(\mathcal{F}))[-d_G] \tag{2}$$

Remark 1. — The collection $(\mathcal{F}_T,(\delta_{\check\alpha})_{\check\alpha\in\Delta_G})$ as above exists, because k is algebraically closed, and one can take $\mathcal{F}_T=(\sqrt{\Omega})^{2\rho}$ for some square root $\sqrt{\Omega}$ of Ω. One has an exact sequence of abelian group schemes $1\to Z\to T\xrightarrow{\prod\check\alpha}\mathbb{G}_m^{\Delta_G}\to 1$, where Z denotes the center of G. So, two choices of the collection $(\mathcal{F}_T,(\delta_{\check\alpha})_{\check\alpha\in\Delta_G})$ are related by a point of $\mathrm{Bun}_Z(k)$ and the associated Whittaker functors are isomorphic up to the automophism of Bun_G given by tensoring with the corresponding Z-torsor.

Remark 2. — When $\mathcal{F}_T$ is fixed, the functor $\mathrm{Whit}_G : \mathrm{D}^{\prec}(\mathrm{Bun}_G)\to\mathrm{D}^{-}(\mathrm{Spec}\, k)$ does not depend, up to isomorphism, on the choice of the isomorphisms $(\delta_{\check\alpha})_{\check\alpha\in\Delta_G}$. That is, for any $(\lambda_{\check\alpha})_{\check\alpha\in\Delta_G}\in(k^*)^{\Delta_G}$, the functors associated to $(\mathcal{F}_T,(\delta_{\check\alpha})_{\check\alpha\in\Delta_G})$ and $(\mathcal{F}_T,(\lambda_{\check\alpha}\delta_{\check\alpha})_{\check\alpha\in\Delta_G})$ are isomorphic. Indeed, the two diagrams $\mathrm{Bun}_G \xleftarrow{\pi} \mathrm{Bun}_N^{\mathcal{F}_T} \xrightarrow{\epsilon} \mathbb{A}^1$ associated to $(\delta_{\check{\alpha}})_{\check{\alpha}\in\Delta_G}$ and $(\lambda_{\check{\alpha}}\delta_{\check{\alpha}})_{\check{\alpha}\in\Delta_G}$ are isomorphic for the following reason. Since k is algebraically closed, $T(k) \to (k^*)^{\Delta_G}$ is surjective. We pick any preimage $\gamma \in T(k)$ of $(\lambda_{\check{\alpha}})_{\check{\alpha}\in\Delta_G}$ and get the automorphism $(\mathcal{F}_B, \zeta) \mapsto (\mathcal{F}_B, \gamma\zeta)$ of $\mathrm{Bun}_N^{\mathcal{F}_T}$, which together with the idendity of Bun_G and $\mathbb{A}^1$ intertwines the two diagrams.

1.1. Whittaker functor for $\mathbb{GL}_n$. — For $i, j \in \mathbb{Z}$ with $i \leq j$ we denote by $\mathcal{N}_{i,j}$ the stack classifying the extensions of Ω^i by Ω^{i+1} ... by Ω^j, i.e. classifying a vector bundle E_{j-i+1} on X with a complete flag of vector subbundles $0 = E_0 \subset E_1 \subset \ldots \subset E_{j-i+1}$ together with isomorphisms $E_{k+1}/E_k \simeq \Omega^{j-k}$ for $k = 0, \ldots, j-i$. Write $\epsilon_{i,j} : \mathcal{N}_{i,j} \to \mathbb{A}^1$ for the map given by the sum of the classes in $\mathrm{Ext}^1(\mathcal{O}, \Omega) \widetilde{\to} \mathbb{A}^1$ of the extensions $0 \to E_{k+1}/E_k \to E_{k+2}/E_k \to E_{k+2}/E_{k+1} \to 0$ for $k = 0, \ldots, j-i-1$.

For $G = \mathbb{GL}_n$, we consider the diagram $\mathrm{Bun}_n \xleftarrow{\pi_{0,n-1}} \mathcal{N}_{0,n-1} \xrightarrow{\epsilon_{0,n-1}} \mathbb{A}^1$, where $\pi_{0,n-1} : \mathcal{N}_{0,n-1} \to \mathrm{Bun}_n$ is $(0 = E_0 \subset \cdots \subset E_n) \mapsto E_n$. This diagram is isomorphic to the diagram $\mathrm{Bun}_G \xleftarrow{\pi} \mathrm{Bun}_N^{\mathcal{F}_T} \xrightarrow{\epsilon} \mathbb{A}^1$ associated to the choice of $\mathcal{F}_T$ whose image in Bun_n is $\Omega^{n-1} \oplus \Omega^{n-2} \oplus \cdots \oplus \mathcal{O}$.

Therefore the functor $\mathrm{Whit}_{\mathbb{GL}_n} : \mathrm{D}^{\prec}(\mathrm{Bun}_n) \to \mathrm{D}^-(\mathrm{Spec}\, k)$ associated to the above choice of $\mathcal{F}_T$ is given by

$$\mathrm{Whit}_{\mathbb{GL}_n}(\mathcal{F}) = \mathrm{R}\Gamma_c(\mathcal{N}_{0,n-1}, \epsilon_{0,n-1}^*(\mathcal{L}_\psi) \otimes \pi_{0,n-1}^*(\mathcal{F}))[\dim \mathcal{N}_{0,n-1} - \dim \mathrm{Bun}_n].$$

REMARK 3. — If E is an irreducible rank n local system on X let Aut_E be the corresponding automorphic sheaf on Bun_n (cf. [2]) normalized to be perverse. Then Aut_E is equipped with a canonical isomorphism $\mathrm{Whit}_{\mathbb{GL}_n}(\mathrm{Aut}_E) \widetilde{\to} \bar{\mathbb{Q}}_\ell$. This is our motivation for the above shift normalization in (2).

1.2. Whittaker functor for $\mathbb{S}\mathrm{p}_{2n}$. — Write G_n for the group scheme on X of automorphisms of $\mathcal{O}^n \oplus \Omega^n$ preserving the natural symplectic form $\wedge^2(\mathcal{O}^n \oplus \Omega^n) \to \Omega$. The stack Bun_{G_n} of G_n-torsors on X can be seen as the stack classifying vector bundles M over X of rank $2n$ equipped with a non-degenerate symplectic form $\Lambda^2 M \to \Omega$.

The diagram $\mathrm{Bun}_{G_n} \xleftarrow{\pi_{G_n}} \mathcal{N}_{G_n} \xrightarrow{\epsilon_{G_n}} \mathbb{A}^1$ constructed in the next definition is isomorphic to the diagram $\mathrm{Bun}_G \xleftarrow{\pi} \mathrm{Bun}_N^{\mathcal{F}_T} \xrightarrow{\epsilon} \mathbb{A}^1$ associated, for $G = G_n$, to the choice of $\mathcal{F}_T$ whose image in Bun_{G_n} is $L \oplus L^* \otimes \Omega$ with $L = \Omega^n \oplus \Omega^{n-1} \oplus \cdots \oplus \Omega$ (with the natural symplectic structure for which L and $L^* \otimes \Omega$ are lagrangians).

DEFINITION 1. — Let $\mathcal{N}_{G_n}$ be the stack classifying $((L_1, \ldots, L_n), E)$, where $(0 = L_0 \subset L_1 \subset \ldots \subset L_n) \in \mathcal{N}_{1,n}$, and E is an extension of $\mathcal{O}_X$-modules

$$0 \to \mathrm{Sym}^2 L_n \to E \to \Omega \to 0 \tag{3}$$