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A NOTE ON INTERSECTIONS OF SIMPLICES

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ABSTRACT. — We provide a corrected proof of [1, Théorème 9] stating that any metrizable infinite-dimensional simplex is affinely homeomorphic to the intersection of a decreasing sequence of Bauer simplices.

RÉSUMÉ (*Sur certaines intersections de simplexes*). — Nous exposons une démonstration rectifiée de [1, Théorème 9], montrant ainsi que tout simplexe de Choquet métrisable et de dimension infinie se représente comme intersection d'une suite décroissante de simplexes de Bauer.

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1. Introduction

If X is a compact convex subset of a locally convex space over the real numbers, it is called a *Choquet simplex* (briefly *simplex*) if the dual $(A(X))^*$ to the space $A(X)$ of all affine continuous functions is a lattice. If, moreover, the set $\operatorname{ext} X$ of all extreme points of X is closed, X is termed a *Bauer simplex* (see [2] for more information on simplices).

The following theorem can be found as [1, Théorème 9]. By (ℓ^1, w^*) we mean ℓ^1 with the topology $\sigma(\ell^1, c_0)$.

THEOREM 1.1. — *Let X be a metrizable infinite-dimensional simplex. Then there exists a decreasing sequence $(T_n)_{n\in\mathbb{N}}$ of Bauer simplices in (ℓ^1, w^*) such that $\bigcap_{n=1}^\infty T_n$ is affinely homeomorphic to X.*

Unfortunately, the proof presented in [1] is not entirely correct, since the inclusion

$$S_{n+1} \cup F_{n+1} \subset \big(\operatorname{conv}(S_n \cup \{e^{n+1}\})\big) \cup F_{n+1}$$

on page 237 of [1] need not hold in general.

The aim of our note is to indicate how to mend the proof of this theorem.

By [3, Theorem 5.2] (see also [2, Theorem 3.22]), for every metrizable infinite-dimensional simplex X there exists an inverse sequence $(X_n, \varphi_n)_{n\in\mathbb{N}}$ of $(n-1)$-dimensional simplices such that X is affinely homeomorphic to its inverse limit $\varprojlim X_n$. More precisely, every $\varphi_n : X_{n+1} \to X_n$ is an affine continuous surjection and X is affinely homeomorphic to

$$\{(x_n) \in \prod_{n=1}^{\infty} X_n : \varphi_n(x_{n+1}) = x_n, n \in \mathbb{N}\}. \tag{1}$$

Inverse sequences $(X_n, \varphi_n)_{n\in\mathbb{N}}$ and $(Y_n, \psi_n)_{n\in\mathbb{N}}$ are called *equivalent* if there exist affine homeomorphisms $\omega_n : X_n \to Y_n$ such that $\omega_n \circ \varphi_n = \psi_n \circ \omega_{n+1}$, $n \in \mathbb{N}$. Clearly, two equivalent inverse sequences have the same inverse limit up to an affine homeomorphism.

A description of a simplex by an inverse sequence yields a method of representing X by an infinite matrix A that is constructed inductively as follows.

In the first step, let $X_1 = \{u_1^1\}$.

Assume now that $n \in \mathbb{N}$ and $\{u_1^n, \dots, u_n^n\}$ is the enumeration of vertices of X_n chosen in the n–th step.

We choose vertices $\{u_1^{n+1}, \dots, u_n^{n+1}\}$ of X_{n+1} such that $\varphi_n(u_i^{n+1}) = u_i^n$, $i = 1, \dots, n$. If $u_{n+1}^{n+1} \in X_{n+1}$ is the remaining vertex, let $a_{1,n}, \dots, a_{n,n}$ be positive numbers with $\sum_{i=1}^n a_{i,n} = 1$ such that

$$\varphi_n(u_{n+1}^{n+1}) = \sum_{i=1}^n a_{i,n} u_i^n.$$

Then

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ 0 & a_{2,2} & a_{2,3} & \cdots \\ 0 & 0 & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the *representing matrix* of X.

It is not difficult to see that A is uniquely determined by the inverse sequence $(X_n, \varphi_n)_{n\in\mathbb{N}}$.

Conversely, any such matrix describes a unique inverse sequence of simplices and thus codes a unique metrizable simplex.

We refer the reader to [2], [3], [4] and [5] for detailed information on representing matrices.

We need the following observation based upon [4, Theorem 4.7].

PROPOSITION 1.2. — *Let A be a representing matrix for a simplex X. Then there exists a matrix $B = \{b_{i,n}\}_{n=1,2,\dots}^{1\le i\le n}$ representing X such that $b_{i,n} > 0$ for all $1 \le i \le n$ and $n = 1, 2, \dots$.*

Proof. — It follows from [4, Theorem 4.7] that two matrices A and B represent the same simplex if $\sum_{n=1}^{\infty}\sum_{i=1}^{n} |a_{i,n} - b_{i,n}| < \infty$. Thus it is enough to slightly perturb the coefficients of A to get the required matrix B. □

2. Proof of Theorem 1.1

We recall some notation from [1]. Let e^n, $n \in \mathbb{N}$, denote the standard basis vectors in ℓ^1 and let $e^0 = 0$.

For $n \in \mathbb{N}$, let $E_n = \operatorname{conv}\{e^0, \dots, e^{n-1}\}$ and let $P_n : \ell^1 \to \ell^1$ be the natural projection on the space spanned by vectors $e^0, \dots, e^{n-1}$, precisely

$$P_n : (x_1, x_2, \dots) \mapsto (x_1, \dots, x_{n-1}, 0, 0, \dots), \quad (x_1, x_2, \dots) \in \ell^1.$$

In particular, P_1 maps ℓ^1 onto e^0.

We state an easy observation needed in the proof of Proposition 2.2.

LEMMA 2.1. — *Let X be a finite-dimensional simplex in a vector space E containing 0 and x be a vector not contained in the linear span of X.*

Then for any y in the relative interior of X there exists $\varepsilon > 0$ such that $y + \varepsilon x \in \operatorname{conv}(X \cup \{x\})$.

Proof. — If y is in the relative interior of X and $0 \in X$, there exists $\varepsilon \in (0,1)$ such that $(1-\varepsilon)^{-1}y \in X$. Then

$$y + \varepsilon x = (1-\varepsilon)\frac{y}{1-\varepsilon} + \varepsilon x \in \operatorname{conv}(X \cup \{x\}),$$

which finishes the proof. □

Now we start with the proof of Theorem 1.1. Given a metrizable simplex X, Proposition 1.2 provides an inverse sequence $(X_n, \varphi_n)_{n\in\mathbb{N}}$ such that X is its inverse limit and the associated representing matrix A has all entries $a_{i,n} > 0$ for all $n \in \mathbb{N}$ and $1 \le i \le n$.

Proposition 2.2. — *Let X be a metrizable infinite-dimensional simplex with a representing matrix A such that $a_{i,n} > 0$ for all $n \in \mathbb{N}$ and $1 \le i \le n$.*

Let $(X_n, \varphi_n)_{n\in\mathbb{N}}$ be the inverse sequence associated with A.

Then there exist $(n-1)$–dimensional simplices $S_n \subset \ell^1$, $n \in \mathbb{N}$, such that

(i) $S_n \subset E_n$, $n \in \mathbb{N}$,
(ii) S_n *is a face of* S_m, $n < m$,
(iii) $P_n S_m = S_n$, $n < m$,
(iv) $S_{n+1} \subset \operatorname{conv}(S_n \cup \{e^n\})$, $n \in \mathbb{N}$,
(v) *the inverse sequences* $(X_n, \varphi_n)_{n\in\mathbb{N}}$ *and* $(S_n, P_n)_{n\in\mathbb{N}}$ *are equivalent.*

Proof. — We construct inductively simplices S_n together with mappings $\omega_n : X_n \to S_n$, $n \in \mathbb{N}$, observing that the resulting inverse sequence is equivalent to the original one.

We start the construction by setting $S_1 = E_1 = \{e^0\}$ and $S_2 = E_2 = \operatorname{conv}\{e^0, e^1\}$. Let $\omega_1 : X_1 \to S_1$ and $\omega_2 : X_2 \to S_2$ be the obvious affine homeomorphisms.

We assume that the construction has been completed up to the n–th stage. If $\omega_n : X_n \to S_n$ is the affine homeomorphism guaranteed by the inductive assumption and $\{u_1^n, \dots, u_n^n\}$ are the vertices of X_n, then $\{\omega_n(u_1^n), \dots, \omega_n(u_n^n)\}$ are the vertices of S_n.

Let $\{u_1^{n+1}, \dots, u_n^{n+1}\}$ be the vertices of X_{n+1} that are mapped by φ_n onto the vertices $\{u_1^n, \dots, u_n^n\}$ of X_n and let u_{n+1}^{n+1} be the remaining vertex mapped onto the point $\sum_{i=1}^n a_{i,n} u_i^n$.

Since all numbers $a_{1,n}, \dots, a_{n,n}$ are strictly positive, the point

$$\omega_n(\varphi_n(u_{n+1}^{n+1})) = \sum_{i=1}^n a_{i,n}\omega_n(u_i^n)$$

is contained in the relative interior of S_n. By Lemma 2.1, there exists $\varepsilon > 0$ such that

$$\omega_n(\varphi_n(u_{n+1}^{n+1})) + \varepsilon e^n \in \operatorname{conv}(S_n \cup \{e^n\}). \tag{2}$$