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MINIMAL SYSTEMS AND DISTRIBUTIONALLY SCRAMBLED SETS

BY PIOTR OPROCHA

ABSTRACT. — In this paper we investigate numerous constructions of minimal systems from the point of view of $(\mathscr{F}_1, \mathscr{F}_2)$-chaos (but most of our results concern the particular cases of distributional chaos of type 1 and 2). We consider standard classes of systems, such as Toeplitz flows, Grillenberger K-systems or Blanchard-Kwiatkowski extensions of the Chacón flow, proving that all of them are DC2. An example of DC1 minimal system with positive topological entropy is also introduced. The above mentioned results answer a few open problems known from the literature.

1. Introduction

The notion of distributional chaos was introduced by Schweizer and Smıtal in 1994 in [29] as a property equivalent to positive topological entropy for maps acting on the unit interval (it extends the notion of pair chaotic in the sense of Li and Yorke, which was know to be not sufficiently strong to imply positive topological entropy). Presently, we have at least three different definitions of distributionally chaotic pair [5] and it was also observed that uniform constant

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of separation of orbits may be important when dealing with this kind of chaos [23]. Recently, Xiong and Tan used in [31] families of subset of integers to define chaotic maps, obtaining that way an interesting and general definition. We adopt this approach here (all the necessary definitions are postponed to the next section).

In [29] the name distributional chaos do not appear explicitly, however it is proved there (in different terminology) that on the unit interval, one DC3 pair is enough to the existence of distributionally ε-scrambled set and both these properties are equivalent to positive topological entropy. It is also interesting that chaos means in [29] the existence of uncountable set whose any two distinct elements form a DC2 pair (so a condition somewhere in the middle between the above two properties). In general setting (i.e. beyond dimension one) there is no more such equivalence, that is, there are systems with positive topological entropy and no DC1 pairs [27] (even minimal ones [4]) as well as systems with DC1 pairs but entropy zero [22].

There are only a few general tools detecting distributionally scrambled sets (e.g. see [30, 3, 25]) however dynamics of systems fulfilling assumptions of these results is highly non-minimal. In the case of minimal maps some methods of construction have been developed, however they have either entropy zero (e.g. see [22, 23, 31]) or do not contain DC1 pairs (e.g. see [4]), while containing plenty of DC2 pairs. In fact, the most challenging conjecture related to distributional chaos (probably first stated by Smítal and then repeated by others, including the author himself) is that every system with positive entropy must contain a DC2 pair.

The main aim of this article is to examine various constructions of minimal systems with positive topological entropy (e.g. Toeplitz flows, extensions of Chacón flow, minimal K-systems, etc.) from the point of view of distributional chaos, or more generally $(\mathscr{F}_1, \mathscr{F}_2)$-chaos, where $\mathscr{F}_1$,$\mathscr{F}_2$ are upward hereditary sets of subsets of $\mathbb{N}$ (so-called Furstenberg families). That way we provide many methods of construction of minimal dynamical systems having uncountably many distributionally chaotic pairs (or not having them at all), filling a gap existing in the literature of the topic and answering a few open problems stated before (e.g. these stated by Balibrea and Smítal in [4]). Especially, two constructions contained in the paper can be of interest: a minimal system with positive entropy and DC1 pairs (see Theorem 9.3) and minimal system with positive entropy but without DC1 pairs nor regularly recurrent points (see Theorem 6.1). Second of this examples follows form a general fact that almost 1-1 extensions of minimal distal systems never have DC1 pairs (see Corollary 4.2). We also prove that every minimal u.p.e. system has plenty of DC2 pairs (see Theorem 7.6), which provides a partial answer (in a very restricted case) to the general conjecture on entropy and DC2 mentioned before.

2. Preliminaries

2.1. Basic notation. — In this paper X is always assumed to be a compact metric space with a metric d, and $f\colon X \to X$ to be continuous. The set of all such maps is denoted $C(X)$. Open balls are denoted by $B(x,\varepsilon) := \{y \in X : d(x,y) < \varepsilon\}$. The same notation is used for every nonempty set $A \subset X$, that is $B(A,\varepsilon) := \bigcup_{x\in A} B(x,\varepsilon)$.

If (X,d_1), (Y,d_2) are metric spaces, we always endow $X \times Y$ with the *maximum metric* $\rho((x,y),(p,q)) = \max\{d_1(x,p), d_2(y,q)\}$. The diagonal in the product $X \times X$ is denoted $\Delta := \{(x,x) : x \in X\}$ and $\Delta_\varepsilon := B(\Delta,\varepsilon)$ for any $\varepsilon > 0$. By a *perfect set* we mean a nonempty compact set without isolated points and by a *Cantor set* we mean a perfect and totally disconnected set. If a set A contains a countable intersection of open and dense subsets of X, then we say that A is *residual* in X.

By $\mathrm{Orb}^+(x)$ we denote the set $\mathrm{Orb}^+(x) := \{x, f(x), f^2(x), \dots\}$ and call it the *(positive) orbit* of a point x. If f is invertible, then we define orbit of x by $\mathrm{Orb}(x) := \{f^i(x) : i \in \mathbb{Z}\}$. A point $y \in X$ is an *ω-limit point* of a point x if it is an accumulation point of the sequence $x, f(x), f^2(x), \dots$. The set of all ω-limit points of x is said to be the *ω-limit set* of x or *positive limit set of* x and is denoted $L^+(x,f)$; we reserve symbol ω to denote another property. We say that a point x is *periodic* if $f^n(x) = x$ for some $n \geq 1$ and *recurrent* if $x \in L^+(x,f)$. Every set M which is nonempty, closed, invariant (i.e. $f(M) \subset M$) and has no proper subset with these three properties is said to be *a minimal set.* If X is the minimal set for f then we say that f is a *minimal system.* Elements of minimal system are usually said to be *minimal points.*

Points $x, y \in X$ are *proximal*, if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$. We say that a point x is *distal*, if it is not proximal to any point $y \in L^+(x,f) \setminus \{x\}$. We say that a nonempty set A is *synchronously proximal* if $\liminf_{n\to\infty} \operatorname{diam} f^n(A) = 0$. It is known that every point is proximal to some minimal point [11] (this statement is nontrivial when given point is not minimal), so distal points are always minimal. We say that f is distal if all of its points are distal (by the above, such a system is always a disjoint sum of minimal systems).

Let X and Y be compact metric spaces and let $f \in C(X)$, $g \in C(Y)$. If there is a continuous onto map $\phi\colon X \to Y$ with $\phi \circ f = g \circ \phi$, we say that f and g are *semiconjugate* (by ϕ). The map ϕ is said to be a *semiconjugacy* (between f and g) or a *factor map*, the map g is said to be a *factor* of f and the map f is said to be an *extension* of g.

2.2. Families and filters. — Here we recall basic facts related to families and filters. Our notation follows [2] together with some concepts from [31].

A *(Furstenberg) family* $\mathscr{F}$ is a collection of subsets of $\mathbb{N}$ which is *upwards hereditary*, that is

$$F_1 \in \mathscr{F} \text{ and } F_1 \subset F_2 \quad \Longrightarrow \quad F_2 \in \mathscr{F}.$$

A family is *proper* if $\mathbb{N} \in \mathscr{F}$ and $\varnothing \notin \mathscr{F}$. Recall that a set $A \subset \mathbb{N}$ is *thick* if for every $n > 0$ there is i such that $\{i, i+1, \ldots, i+n\} \subset A$. We denote by $\mathscr{B}$ the family of infinite subsets of $\mathbb{N}$ and by $\tau\mathscr{B}$ the family of all thick subsets of $\mathbb{N}$.

For every family $\mathscr{F}$ we define its *dual family* by $k\mathscr{F} := \{F \subset \mathbb{N} \,:\, \mathbb{N} \setminus F \notin \mathscr{F}\}$. Elements of the dual family $k\tau\mathscr{B}$ are said to be *syndetic sets*.

If $\mathscr{F}_1$ and $\mathscr{F}_2$ are families then we define

$$\mathscr{F}_1 \cdot \mathscr{F}_2 := \{F_1 \cap F_2 \,:\, F_1 \in \mathscr{F}_1, F_2 \in \mathscr{F}_2\}.$$

Note that $\mathscr{F}_1 \cup \mathscr{F}_2 \subset \mathscr{F}_1 \cdot \mathscr{F}_2$ for any two proper families $\mathscr{F}_1, \mathscr{F}_2$. We say that families $\mathscr{F}_1$ and $\mathscr{F}_2$ meet when $\mathscr{F}_1 \cdot \mathscr{F}_2$ is proper. Let $\mathscr{P}(\mathbb{N})$ denote the power set of $\mathbb{N}$. For $\mathscr{A} \subset \mathscr{P}(\mathbb{N})$ we define *the family generated by* $\mathscr{A}$ as

$$[\mathscr{A}] := \{F \subset \mathbb{N} \,:\, A \subset F \text{ for some } A \in \mathscr{A}\}$$

For the case $\mathscr{A} = \{A\}$, where $A \subset \mathbb{N}$, we simply write $[A]$ instead of $[\mathscr{A}]$. For any infinite set $Q \in \mathscr{B}$ we define $\langle Q \rangle := [T(Q)]$ where $T(Q)$ is the set of tails of Q, i.e. $T(Q) = \{Q \cap [n, +\infty) \,:\, n \in \mathbb{N}\}$.

A *filter* $\mathscr{F}$ is a proper family such that $\mathscr{F} \cdot \mathscr{F} = \mathscr{F}$. Let $\mathscr{A} \subset \mathscr{P}(\mathbb{N})$ and denote $\mathscr{A}^\cap := \{A_1 \cap \cdots \cap A_n \,:\, A_i \in \mathscr{A}, n > 0\}$. If $\varnothing \notin \mathscr{A}^\cap$ then $[\mathscr{A}^\cap]$ is a filter. In that case we say that $\mathscr{A}$ *generates a filter* and call $[\mathscr{A}^\cap]$ *the filter generated by* $\mathscr{A}$.

Note that, if $Q \in \mathscr{B}$ then $[Q]$ and $\langle Q \rangle$ are filters generated by $\{Q\}$ and $T(Q)$ respectively, since $\{Q\}^\cap = \{Q\}$ and $T(Q)^\cap = T(Q)$.

Given $A \subset X$ and $x \in X$ we write $N(x, A, f) = \{n : f^n(x) \in A\}$. If A, B are sets then $N(A, B, f) = \{n : f^{-n}(B) \cap A \neq \varnothing\}$. We say that a point x is *uniformly recurrent* if the set $N(x, U, f)$ is syndetic for every open set $U \ni x$. It is known that every element of a minimal set is uniformly recurrent, and $L^+(x, f)$ is a minimal set for every uniformly recurrent point (this was first proved by Birkhoff), in particular minimal or uniformly recurrent points define the same property in different language.

Let $P = \{p_1 < p_2 < \cdots\} \in \mathscr{B}$. Define

$$\mathscr{D}^*(P) := \limsup_{n\to\infty} \frac{\#(P \cap \{1, \ldots, n\})}{n} \quad , \quad \mathscr{D}_*(P) := \liminf_{n\to\infty} \frac{\#(P \cap \{1, \ldots, n\})}{n}.$$

We say that $\mathscr{D}^*(P)$ and $\mathscr{D}_*(P)$ are *the upper density* and *the lower density of* P respectively. If $\mathscr{D}^*(P) = \mathscr{D}_*(P)$ then we denote $\mathscr{D}(P) := \mathscr{D}^*(P)$ and call this number *the density of* P.