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DIOPHANTINE APPROXIMATION ON VEECH SURFACES

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ABSTRACT. — We show that Y. Cheung's general Z-continued fractions can be adapted to give approximation by saddle connection vectors for any compact translation surface. That is, we show the finiteness of his Minkowski constant for any compact translation surface. Furthermore, we show that for a Veech surface in standard form, each component of any saddle connection vector dominates its conjugates in an appropriate sense. The saddle connection continued fractions then allow one to recognize certain transcendental directions by their developments.

RÉSUMÉ (*Approximation diophantienne sur les surfaces de Veech*)

Nous montrons que les fractions continues generalisées Z de Y. Cheung s'adaptent pour exprimer l'approximation par vecteurs de connexion de selles sur n'importe quelle surface de translation compacte. C'est-à-dire, nous démontrons la finitude de la constant de Minkowski pour chaque surface de translation compacte. De plus, pour une surface de Veech en forme standard, nous montrons que chaque composant de n'importe quel vecteur de connexion de selle domine, dans un sens approprié, ses conjugués. Les fractions continues de connexions de selle permettent de reconnaître certaines directions transcendantales par leur développement.

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1. Introduction and Main Results

We show that Yitwah Cheung's generalization of the geometric interpretation of regular continued fractions gives a successful method for approximation of flow directions on translation surfaces by saddle connection vectors. Cheung [7], [8] generalizes the work of Poincaré and Klein by replacing approximation by the integer lattice in $\mathbb{R}^2$ with approximation by any infinite discrete set Z of nonzero vectors with finite "Minkowski constant", equal to one-fourth times the supremum taken over the areas of centro-symmetric bounded convex bodies disjoint from Z.

We prove, as Cheung certainly understood, that the set of saddle connection vectors of any translation surface has a finite Minkowski constant.

THEOREM 1. — *Let S be a compact translation surface, and $Z = \mathrm{V_{sc}}(S)$ the set of saddle connection vectors of S. Then*

$$\mu(Z) \leq \pi \operatorname{vol}(S)$$

where $\operatorname{vol}(S)$ *is the Lebesgue area of S and $\mu(Z)$ is as given in Definition 2.*

The following result is of independent interest; here, it allows us to reach transcendence results using approximation by saddle connection vectors. Recall that the group of linear parts (the so-called "derivatives") of the oriented affine diffeomorphisms of a compact finite genus translation surface, S, form a Fuchsian group, $\Gamma(S)$. The trace field of the surface is the algebraic number field generated over the rationals by the set of traces of the elements of $\Gamma(S)$, when this group is non-trivial. When $\Gamma(S)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$, the surface is said to be a Veech surface.

THEOREM 2. — *Suppose that S is a Veech surface normalized so that:* $\Gamma(S) \subset \mathrm{SL}_2(\mathbb{K})$*; the horizontal direction is periodic; and, both components of every saddle connection vector of S lie in $\mathbb{K}$, where $\mathbb{K}$ is the trace field of S. Then there exists a positive constant $c = c(S)$ such that for all holonomy vectors $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and $1 \leq i \leq 2$ one has*

$$|\, v_i \,| \geq c \,|\, \sigma(\, v_i) \,|\,,$$

where σ varies through the set of field embeddings of $\mathbb{K}$ into $\mathbb{R}$.

Note that in the above, each field embedding σ in fact takes values only in $\mathbb{R}$.

With S as above and $Z = \mathrm{V_{sc}}(S)$ the set of saddle connection vectors of S, the Z-expansion of an inverse slope θ for a flow direction is defined in Section 2.1. Theorem 1 then implies that this gives a sequence of elements

$(p_n, q_n) \in \mathbb{K}^2$ such that $|\theta - p_n/q_n|$ goes to zero as n tends to infinity; see Lemma 2.

One criterion for a "good" continued fraction algorithm is that extremely rapid convergence to a real number implies that this number is transcendental. We show that the $Z = \mathrm{V}_{\mathrm{sc}}(S)$-fractions on Veech surfaces enjoy this property.

Theorem 3. — *With S and $\mathbb{K}$ as above, let $D = [\mathbb{K} : \mathbb{Q}]$ be the field extension degree of $\mathbb{K}$ over the field of rational numbers. If a real number $\xi \in [0,1] \setminus \mathbb{K}$ has an infinite $\mathrm{V}_{\mathrm{sc}}(S)$-expansion, whose convergents p_n/q_n satisfy*

$$\limsup_{n\to\infty} \frac{\log\log q_n}{n} > \log(\,2D - 1\,)\,,$$

then ξ is transcendental.

1.1. Related work. — There exist algorithms that approximate flow directions on particular translation surfaces by so-called parabolic directions, see [1], [25], [24]. Roughly speaking, these algorithms can be viewed as continued fraction algorithms expressing real values in terms of the orbit of infinity under the action of a related Fuchsian group. Up to finite index and appropriate normalization, each underlying group in these examples is one of the infinite family of Hecke triangle Fuchsian groups, [28]. Some 60 years ago, for each Hecke group, D. Rosen [21] gave a continued fraction algorithm. Motivated in part by the use in [2] of the Rosen fractions to identify pseudo-Anosov directions with vanishing so-called SAF-invariant, with Y. Bugeaud, in [5] we recently gave the first transcendence results using Rosen continued fractions. Theorem 3 is the analog of a main result there.

Each Hecke group is contained in a particular $\mathrm{PSL}(2, K)$ with K a totally real number field. Key to the approach of [5] was the fact that any element in a Hecke group of sufficiently large trace is such that this trace is appropriately larger than each of its conjugates over $\mathbb{Q}$. This leads to a bound of the height of a convergent p_n/q_n in terms of q_n itself. The LeVeque form of Roth's Theorem, in combination with a bound on approximation in terms of $q_n q_{n+1}$, can then be used to show that transcendence is revealed by exceptionally high rates of growth of the q_n. We show here that all of this is possible for any Veech surface, replacing Rosen fractions by Z-expansions with $Z = \mathrm{V}_{\mathrm{sc}}(S)$. Key to this is our results that: (1) any nontrivial Veech group $\Gamma(S)$ has the property of the dominance of traces over their conjugates; and, (2) in the case of a Veech surface S, the dominance property for the group implies a (weaker) dominance of components of saddle vectors over their conjugates.

We mention that it would be interesting to compare the approximation in terms of saddle connection vectors with the known instances of approximation with parabolic directions.

1.2. Outline. — In the following section we sketch some of the disparate background necessary for our results; in Section 3 we prove the crucial result that the Minkowski constant is finite for any compact translation surface; in Section 4 we show that if S is a Veech surface then $\Gamma(S)$ has the property of dominating conjugates and from this that one can bound the heights in the Z-expansions, $Z = \mathrm{V_{sc}}(S)$; finally, in Section 5 we very briefly show that the arguments of [5] are valid here: $Z = \mathrm{V_{sc}}(S)$-expansions with extremely rapidly growing denominators belong to transcendental numbers.

1.3. Thanks. — It is a pleasure to thank Curt McMullen for asking if the results of [5] could hold in the general Veech surface setting. We also thank Emmanuel Russ for pointing out the reference [3]. Finally, we thank the referee for a careful reading and for the suggestion of Corollary 2.

2. Background

2.1. Cheung's Z-expansions. — We briefly review Yitwah Cheung's definition of his Z-expansions — we follow Section 3 of [8], although we focus on approximation of a ray instead of a line. (This simplification is valid in our setting, as we can and do assume that the approximating set Z is symmetric about the origin.)

Fix a discrete set $Z \subset \mathbb{R}^2$, and assume that Z does not contain the zero vector. Given a positive real θ, consider the ray emitted from the origin with slope $1/\theta$. Our goal is to define a sequence of elements of Z that approximates this ray.

Remark 1. — Note that the number that is approximated here is the *inverse of the slope* of the ray. This choice accords well with the *projective* action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

Let u be the unit vector in the direction of the ray. Denote the positive half plane of the ray by $H_+(\theta) = \{v \in \mathbb{R}^2 \mid u{\cdot}v > 0\}$, and let $Z_+(\theta) := Z \cap H_+(\theta)$. Let $v = (p, q) \in \mathbb{R}^2$; the difference vector between v and the vector whose endpoint is given by the intersection of $y = x/\theta$ and $y = q$ has length of absolute value $\mathrm{hor}_\theta(v) = |q\theta - p|$. The value q is the *height* of v and $\mathrm{hor}_\theta(v)$ is its *horizontal component*, see Figure 1.

Definition 1. — The Z*-convergents* of θ is the set of elements of Z in the half-plane of the ray such that each minimizes the horizontal component $\mathrm{hor}_\theta(v)$ amongst elements of equal or lesser height:

$$\mathrm{Conv}_Z(\theta) = \{v \in Z_+(\theta) \mid \forall w \in Z_+(\theta), |w_2| \le |v_2| \implies \mathrm{hor}_\theta(v) \le \mathrm{hor}_\theta(w)\}.$$