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ASYMPTOTIC VASSILIEV INVARIANTS FOR VECTOR FIELDS

by Sebastian Baader & Julien Marché

Abstract. — We analyse the asymptotical growth of Vassiliev invariants on non-periodic flow lines of ergodic vector fields on domains of $\mathbb{R}^3$. More precisely, we show that the asymptotics of Vassiliev invariants is completely determined by the helicity of the vector field.

Résumé (*Invariants de Vassiliev asymptotiques des champs de vecteurs*)
Nous analysons le comportement asymptotique des invariants de Vassiliev des orbites non périodiques d'un champ de vecteurs ergodique dans un domaine de $\mathbb{R}^3$. Nous montrons que ce comportement est gouverné par l'hélicité du champ de vecteurs.

1. Introduction

A smooth vector field on a manifold defines a flow whose orbits may be closed or not. If the manifold is a compact domain $G \subset \mathbb{R}^3$, we may ask about the asymptotical growth of knot invariants on non-periodic orbits. A well-known and classical example for this is the helicity of a vector field, which measures how pairs of non-periodic orbits are asymptotically linked, in the average [1]. In order to make quantitative statements, we suppose that the flow of the

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vector field X be measure-preserving and ergodic with respect to a probability measure μ on G, further, that the singularities of X be isolated and that the periodic orbits of X be not charged by the flow. For every non-periodic point $p \in G$ and $T > 0$ we define a set $K(p,T) \subset \mathbb{R}^3$, as follows:

$$K(p,T) = \{\phi^X(p,t) | t \in [0,T]\} \cup [p, \phi^X(p,T)],$$

where ϕ^X is the flow of X and $[p, \phi^X(p,T)]$ the geodesic segment in $\mathbb{R}^3$ joining p and $\phi^X(p,T)$. This set is actually a knot, i.e. an embedded circle, for almost all $p \in G$, $T > 0$ ([3], [8]). Under the above hypotheses, Gambaudo and Ghys proved the existence of an asymptotic signature invariant which is proportional to the helicity of X [3]: for almost all $p \in G$ the limit

$$\sigma(X) = \lim_{T \to \infty} \frac{1}{T^2} \sigma(K(p,T)) \in \mathbb{R}$$

exists and is independent of the starting point $p \in G$. Here σ denotes the signature invariant of links. The asymptotic signature invariant determines the asymptotical behaviour of a large class of concordance invariants [2]. In this note we show that Vassiliev invariants are asymptotically determined by the signature (hence also by the helicity).

THEOREM 1. — *Let v be a real-valued Vassiliev knot invariant of degree n. There exists a constant $\alpha_v \in \mathbb{R}$, such that for almost all $p \in G$ the limit*

$$\lim_{T \to \infty} \frac{1}{T^{2n}} v(K(p,T)) \in \mathbb{R}$$

exists and coincides with $\alpha_v \sigma(X)^n$. The constant α_v does not depend on the vector field X.

Gambaudo and Ghys provided the first instance of this theorem since the helicity can be defined as an asymptotical linking number, which is a Vassiliev invariant of degree one (for links, however). The proof of Theorem 1 is based on a asymptotical count of Gauss diagrams with respect to suitable diagrams of the knots $K(p,T)$. We give a short summary of Gambaudo and Ghys' construction in Section 2. The proof of Theorem 1 is contained in Section 3.

REMARK. — For reasons of simplicity, we restrict ourselves to the study of asymptotical knots and their invariants, rather than links. The case of links does not pose any additional difficulties.

2. Asymptotic Diagrams

The main part of Gambaudo and Ghys' work [3] consists in constructing good diagrams for the knots $K(p,T)$. For this purpose, they cover the domain G, away from the singularities of X, by a countable family of flow boxes $\{\mathscr{F}_i\}_{i\in\mathbb{N}}$. Further, they define a projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ onto a plane which is well-adapted to this family: for every $\epsilon > 0$, there exists a finite subset $\mathscr{C} \subset \mathbb{N}$, such that for almost all $p \in G$, $T > 0$ large enough, the diagram $\pi(K(p,T))$ is regular and, up to an error $\leq \epsilon T^2$, its crossings arise from pairs of overcrossing flow boxes $\mathscr{F}_i$, $\mathscr{F}_j$, with $i, j \in \mathscr{C}$. Moreover, at these finitely many overcrossing spots the diagram looks like a rectangular grid, as sketched in Figure 1. We will shortly see that the number of crossings of these grids grows like T^2.

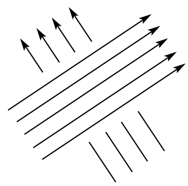

FIGURE 1.

Let $n_i(p,T) = \pi_0(\mathscr{F}_i \cap \{\phi^X(p,t) | t \in [0,T]\})$ be the number of times the flow line starting at p and ending at $\phi^X(p,T)$ enters the flow box $\mathscr{F}_i$. Applying Birkhoff's ergodic theorem to the characteristic function of the flow box $\mathscr{F}_i$, we immediately see that for almost all $p \in G$ the limit

$$n_i = \lim_{T\to\infty} \frac{1}{T} n_i(p,T) > 0$$

exists (and is proportional to the volume of the flow box $\mu(\mathscr{F}_i)$). Therefore the number of crossings $c_{ij}(p,T)$ at an overcrossing spot of two flow boxes $\mathscr{F}_i$, $\mathscr{F}_j$ satisfies

$$\lim_{T\to\infty} \frac{1}{T^2} c_{ij}(p,T) = n_i n_j. \tag{1}$$

For later purposes, we choose a natural number $N \in \mathbb{N}$ and subdivide the time interval $[0,T]$ into N sub-intervals $I_1, I_2, \ldots, I_N$ of length $\frac{T}{N}$. Every index $k \in \{1, 2, \ldots, N\}$ gives rise to a function $n_{i,k}(p,T) = \pi_0(\mathscr{F}_i \cap \{\phi^X(p,t) | t \in I_k\})$. Again, by Birkhoff's theorem, we obtain

$$\lim_{T\to\infty} \frac{1}{T} n_{i,k}(p,T) = \frac{n_i}{N}. \tag{2}$$

At last, for two flow box indices $i_1, i_2 \in \mathscr{C}$, and $k_1, k_2 \in \{1, 2, \ldots, N\}$, we define the number of crossings $c_{i_1,k_1,i_2,k_2}(p,T)$ between $\mathscr{F}_{i_1} \cap \{\phi^X(p,t) | t \in I_{k_1}\})$ and

$\mathcal{F}_{i_2} \cap \{\phi^X(p,t) | t \in I_{k_2}\})$ at an overcrossing spot of the flow boxes $\mathcal{F}_{i_1}$, $\mathcal{F}_{i_2}$. Equation (2) implies

$$\lim_{T\to\infty} \frac{1}{T^2} c_{i_1,k_1,i_2,k_2}(p,T) = \frac{n_{i_1} n_{i_2}}{N^2} \tag{3}$$

This equality will play an important role in the proof of Theorem 1.

3. Gauss Diagram Formulae and Proof of Theorem 1

A Gauss diagram is nothing but a special notation for a knot diagram. It consists of an oriented circle with a finite number of signed arrows connecting pairs of points on the circle. The circle stands for the oriented knot itself, while the arrows encode crossing points of the knot diagram, pointing from the lower to the upper strand. Their signs indicate the signs of their crossings. For example, Figure 2 shows a Gauss diagram representing the standard diagram of the twist knot with six crossings. Here the orientation of the circle is understood to be clockwise.

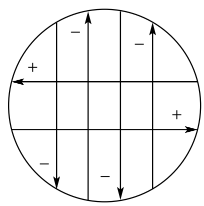

FIGURE 2.

It is often convenient to consider pointed Gauss diagram, i.e. Gauss diagrams with a distinguished base point on its circle. Throughout this section, we will work with pointed Gauss diagram. In particular, we will be concerned with the pointed Gauss diagrams $G(p,T)$ arising from Gambaudo and Ghys' special diagrams $D(p,T) = \pi(K(p,T))$.

Gauss diagrams are of special interest in the theory of Vassiliev invariants, since the latter can be identified with certain formal linear combinations of Gauss diagrams. In order to explain this, we have to introduce a pairing between Gaus diagrams. Let Γ, G be two Gauss diagrams. The expression

$$\langle \Gamma, G \rangle$$

is defined as the weighted number of sub-diagrams of G isomorphic to Γ, respecting the circles, base points and all orientations. The weights are simply the products over all signs of arrows of the corresponding subgraphs and Γ.