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SOME REMARKS ON THE LOCAL CLASS FIELD THEORY OF SERRE AND HAZEWINKEL

BY TAKASHI SUZUKI

ABSTRACT. — We give a new approach for the local class field theory of Serre and Hazewinkel. We also discuss two-dimensional local class field theory in this framework.

RÉSUMÉ (*Quelques remarques sur la théorie du corps de classes local de Serre et Hazewinkel*)

Nous donnons une nouvelle approche de la théorie du corps de classes local de Serre et Hazewinkel. Nous discutons également la théorie du corps de classes local de dimension deux dans ce cadre.

1. Introduction

The purpose of this paper is twofold. First, we give a geometric description of the local class field theory of Serre and Hazewinkel ([13], [7, Appendice]) in the equal characteristic case. The main result is Theorem A below. Second, we discuss its two-dimensional analog with the aim to seek an analog of Lubin-Tate theory for two-dimensional local fields. The discussion is taken place in Section 6 with a partial result.

We formulate Theorem A. The precise definitions of the terms used below are given in Section 2. Let k be a perfect field of characteristic $p > 0$ and set $K = k((T))$. The group of units of K can be viewed as a proalgebraic group

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(more precisely, a pro-quasi-algebraic group) over k in the sense of Serre ([12]); we denote this proalgebraic group by $\mathbf{U}_K$. For each perfect k-algebra R (perfect means that the p-th power map is an isomorphism), we have the group of its R-rational points

$$\mathbf{U}_K(R) = \left\{ \sum_{i=0}^{\infty} a_i \mathbf{T}^i \;\middle|\; a_i \in R,\ a_0 \in R^\times \right\}.$$

Likewise, the multiplicative group $K^\times$ of K can be viewed as a perfect group scheme $\mathbf{K}^\times$, which is the direct product of $\mathbf{U}_K$ and the discrete infinite cyclic group generated by $\mathbf{T}$. Let K_p be the perfect closure of K. We consider the K_p-rational point $1 - T\mathbf{T}^{-1}$ of $\mathbf{K}^\times$ and the corresponding morphism

$$\varphi \colon \operatorname{Spec} K_p \to \mathbf{K}^\times.$$

This morphism induces a homomorphism

$$\eta \colon \operatorname{Gal}(K^{\mathrm{ab}}/K) = \pi_1^{\mathrm{et}}(\operatorname{Spec} K_p)^{\mathrm{ab}} \xrightarrow{\varphi} \pi_1^k(\mathbf{K}^\times)$$

on the fundamental groups. Here K^{ab} is the maximal abelian extension of K and $\pi_1^{\mathrm{et}}(\cdot)^{\mathrm{ab}}$ denotes the maximal abelian quotient of the étale fundamental group. The group $\pi_1^k(\mathbf{K}^\times)$ is the fundamental group of $\mathbf{K}^\times$ as a perfect group scheme over k, which classifies all surjective isogenies to $\mathbf{K}^\times$ with finite constant kernels. Now we state the main theorem of this paper:

Theorem A. — 1. *The above defined map* $\eta \colon \operatorname{Gal}(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{K}^\times)$ *is an isomorphism.*

2. *The inverse of* η *restricted to* $\pi_1^k(\mathbf{U}_K)$ *coincides with the reciprocity isomorphism* $\theta \colon \pi_1^k(\mathbf{U}_K) \xrightarrow{\sim} I(K^{\mathrm{ab}}/K)$ *of Serre and Hazewinkel ([13], [7]), where* I *denotes the inertia group.*

Note that Assertion 1 of the theorem says that we have an essentially one-to-one correspondence between surjective isogenies $A \twoheadrightarrow \mathbf{K}^\times$ with finite constant kernels and finite abelian extensions L of K by pullback by φ, as expressed as a cartesian diagram of the form

$$\begin{array}{ccc} \operatorname{Spec} L_p & \longrightarrow & A \\ \downarrow & & \downarrow \\ \operatorname{Spec} K_p & \xrightarrow{\varphi} & \mathbf{K}^\times. \end{array}$$

In other words, we have $\operatorname{Ext}_k^1(\mathbf{K}^\times, N) \xrightarrow{\sim} H^1(K, N)$ for any finite constant group N. Another remark is that the proof of Assertion 2 that we give in this paper does not use the fact that θ is an isomorphism. This means that Theorem A and our proof of it together give another proof of this fact, more specifically, the existence theorem of the local class field theory of Serre and

Hazewinkel ([13, §4], [7, 6.3]). For a generalization of Assertion 2 for the full groups $\pi_1^k(\mathbf{K}^\times)$ and $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, see Remark 3.5.

We outline the paper. After providing some preliminaries at Section 2, we give three different proofs of Theorem A. Each of them has its own advantages and interesting points. The first proof given in Section 3 may be the standard one. The method is purely local. In this proof, for Assertion 1, we explicitly calculate the groups $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ and $\pi_1^k(\mathbf{K}^\times)$ and the homomorphism η between them individually. For Assertion 2, we interpret the assertion to the compatibility of φ with norm maps and prove it by analyzing the diagonal divisors. The second proof given in Section 4 relies on Lubin-Tate theory. Hence it is applicable only for finite residue field cases. The third proof given in Section 5 is a geometric proof, which was suggested by the referee to the author. In this geometric proof, for Assertion 1, we need the Albanese property of the morphism $\varphi\colon \operatorname{Spec} K_p \to \mathbf{K}^\times$, which was established by Contou-Carrere in [6] (see also [5]). For Assertion 2, we need the local-global compatibility and the global version of Assertion 2, which was obtained by Serre in [13, §5].

In Section 6, we define a morphism analogous to $\varphi\colon \operatorname{Spec} K_p \to \mathbf{K}^\times$ for a field of the form $k((S))((T))$ using the second algebraic K-group instead of the multiplicative group. If k is a finite field, the field $k((S))((T))$ is called a two-dimensional local field ([8]) of positive characteristic. In analogy with the second proof of Theorem A by Lubin-Tate theory, we attempt to formulate an analogous theory to Lubin-Tate theory for the two-dimensional local field $k((S))((T))$. Although we have not yet obtained an analog of a Lubin-Tate formal group in this paper, we did obtain a result that may be thought of as a partial result for abelian extensions having p-torsion Galois groups (Proposition 6.1). In Section 7, we give an analog of the result of [2, §2.6] on $\mathcal{D}$-modules. Their result is for fields of the form $k((T))$ with k characteristic zero while ours $k((S))((T))$ with k characteristic zero. This is also regarded as an analog of Proposition 6.1.

We give a couple of comments on literature. The morphism $\varphi\colon \operatorname{Spec} K_p \to \mathbf{K}^\times$ had been introduced by Grothendieck ([4, August 9, 1960]) long before our paper. Besides Contou-Carrere as mentioned above, the morphism φ has been studied by many people. The $\mathcal{D}$-module version of Theorem A in the zero-characteristic case had been established in [2, §2.6]. Deligne had given a sketch of proof of results stronger than Assertion 1. This is written in Section e of his letter to Serre contained in [3]. His method is different from our method. The author did not realize these works at the time of writing this paper.

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2. Preliminaries

In this section, we give precise definitions of the terms that we used in Introduction and will use in the following sections, fix notation, prove the independence of the choice of the prime element T (Proposition 2.3), and reduce the proof of Theorem A to the case of algebraically closed residue fields.

2.1. Definitions and notation. — We work on the site $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ of perfect schemes over a perfect field k of characteristic $p > 0$ with the fpqc topology (compare with [11, Chapter III, §0, "Duality for unipotent perfect group schemes"]; see also [17]). The category of sheaves of abelian groups on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ contains the category of commutative affine pro(-quasi)-algebraic groups over k in the sense of Serre ([12]) as an thick abelian full subcategory. We denote by Ext_k^i the i-th Ext functor for the category of sheaves of abelian groups on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$. For a sheaf of abelian groups A on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ and a non-negative integer i, we define the fundamental group $\pi_1^k(A)$ of A to be the Pontryagin dual of the torsion abelian group $\operatorname{inj\,lim}_{n \geq 1} \mathrm{Ext}_k^1(A, n^{-1}\mathbb{Z}/\mathbb{Z})$. If A is an extension of an étale group whose group of geometric points is finitely generated as an abelian group by an affine proalgebraic group, then $\operatorname{inj\,lim}_{n \geq 1} \mathrm{Ext}_k^1(A, n^{-1}\mathbb{Z}/\mathbb{Z}) = \mathrm{Ext}_k^1(A, \mathbb{Q}/\mathbb{Z})$ and hence the Pontryagin dual of $\mathrm{Ext}_k^1(A, \mathbb{Q}/\mathbb{Z})$ coincides with $\pi_1^k(A)$. For a k-algebra R, we denote by R_p the perfect k-algebra given by the injective limit of p-th power maps $R \to R \to \cdots$, where the i-th copy of R in this system is given the map $k \to R$, $a \mapsto a^{p^i}$ as its k-algebra structure map.

Let K be a complete discrete valuation field of equal characteristic with residue field k. We denote by $\mathcal{O}_K$ the ring of integers of K and by $\mathfrak{p}_K$ the maximal ideal of $\mathcal{O}_K$. We set $U_K = U_K^0 = \mathcal{O}_K^\times$ and $U_K^n = 1 + \mathfrak{p}_K^n$ for $n \geq 1$. We fix an algebraic closure $\overline{K}$ of K. All algebraic extensions of K are taken inside $\overline{K}$. We denote by K^{sep} ($\subset \overline{K}$) the separable closure of K, by K^{ur} the maximal unramified extension of K and by K^{ab} the maximal abelian extension of K, respectively. Since K has equal characteristic $p > 0$, the rings $\mathcal{O}_K$ and K have canonical structures of k-algebras by the Teichmüller section $k \hookrightarrow \mathcal{O}_K$. Hence, for a sheaf of abelian groups A on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$, we can consider the groups $A((\mathcal{O}_K)_p)$ (resp. $A(K_p)$) of $(\mathcal{O}_K)_p$-rational (resp. K_p-rational) points of A. We denote by $A((\mathfrak{p}_K)_p)$ the kernel of the natural map $A((\mathcal{O}_K)_p) \to A(k)$. The reduction map $\mathcal{O}_K \twoheadrightarrow k$ and the Teichmüller section $k \hookrightarrow \mathcal{O}_K$ give a canonical splitting $A((\mathcal{O}_K)_p) = A(k) \oplus A((\mathfrak{p}_K)_p)$.