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A RIEMANN-ROCH THEOREM FOR DG ALGEBRAS

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ABSTRACT. — Given a smooth proper dg algebra A, a perfect dg A-module M and an endomorphism f of M, we define the Hochschild class of the pair (M, f) with values in the Hochschild homology of the algebra A. Our main result is a Riemann-Roch type formula involving the convolution of two such Hochschild classes.

RÉSUMÉ (*Un théorème de Riemann-Roch pour les dg algèbres*). — Étant donnée une dg algèbre A, propre et lisse, un dg A-module parfait M et un endomorphisme f de M, nous définissons la classe de Hochschild de la paire (M, f). Cette classe est à valeurs dans l'homologie de Hochschild de l'algèbre A. Notre principal résultat est une formule de type Riemann-Roch faisant intervenir la convolution de deux de ces classes de Hochschild.

1. Introduction

An algebraic version of the Riemann-Roch formula was recently obtained by D. Shklyarov [26] in the framework of the so-called noncommutative derived algebraic geometry. More precisely, motivated by the well known result of A. Bondal and M. Van den Bergh about "dg-affinity" of classical varieties, D.

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Shklyarov has obtained a formula for the Euler characteristic of the Hom-complex between two perfect modules over a dg-algebra in terms of the Euler classes of the modules.

On the other hand, M. Kashiwara and P. Schapira [12] initiated an approach to the Riemann-Roch theorem in the framework of deformation quantization modules (DQ-modules) with the view towards applications to various index type theorems. Their approach is based on Hochschild homology which, in this setup, admits a description in terms of the dualizing complexes in the derived categories of coherent DQ-modules.

The present paper is an attempt to extract some algebraic aspects of this latter approach with the hope that the resulting techniques will provide a uniform point of view for proving Riemann-Roch type results for DQ-modules, D-modules etc. (e.g. the Riemann-Roch-Hirzebruch formula for traces of differential operators obtained by M. Engeli and G. Felder [6]). In this paper, we obtain a Riemann-Roch theorem in the dg setting, similarly as D. Shklyarov. However, our approach is really different of the latter one in that we avoid the categorical definition of the Hochschild homology, and use instead the Hochschild homology of the ring A expressed in terms of dualizing objects. Our result is slightly more general than the one obtained in [26]. Instead of a kind of non-commutative Riemann-Roch theorem, we rather prove a kind of non-commutative Lefschetz theorem. Indeed, it involves certain Hochschild classes of *pairs* (M, f) where M is a perfect dg module over a smooth proper dg algebra and f is an endomorphism of M in the derived category of perfect A-modules. Moreover, our approach follows [12]. In particular, we have in our setting relative finiteness and duality results (Theorem 3.16 and Theorem 3.27) that may be compared with [12, Theorem 3.2.1] and [12, Theorem 3.3.3]. Notice that the idea to approach the classical Riemann-Roch theorem for smooth projective varieties via their Hochschild homology goes back at least to the work of N. Markarian [17]. This approach was developed further by A. Căldăraru [3], [4] and A. Căldăraru, S. Willerton [5] where, in particular, certain purely categorical aspects of the story were emphasized. The results of [3] suggested that a Riemann-Roch type formula might exist for triangulated categories of quite general nature, provided they possess Serre duality. In this categorical framework, the role of the Hochschild homology is played by the space of morphisms from the inverse of the Serre functor to the identity endofunctor. In a sense, our result can be viewed as a non-commutative generalization of A. Căldăraru's version of the topological Cardy condition [3]. Our original motivation was different though it also came from the theory of DQ-modules [12].

Here is our main result:

THEOREM. — *Let A be a proper, homologically smooth dg algebra, $M \in \mathsf{D}_{\mathrm{perf}}(A)$, $f \in \mathrm{Hom}_A(M,M)$ and $N \in \mathsf{D}_{\mathrm{perf}}(A^{\mathrm{op}})$, $g \in \mathrm{Hom}_{A^{\mathrm{op}}}(N,N)$.*

Then

$$\mathrm{hh}_k(N \overset{\mathrm{L}}{\underset{A}{\otimes}} M, g \overset{\mathrm{L}}{\underset{A}{\otimes}} f) = \mathrm{hh}_{A^{\mathrm{op}}}(N,g) \cup \mathrm{hh}_A(M,f),$$

where $\cup$ is a pairing between the corresponding Hochschild homology groups and where $\mathrm{hh}_A(M,f)$ is the Hochschild class of the pair (M,f) with value in the Hochschild homology of A.

The above pairing is obtained using Serre duality in the derived category of perfect complexes and, thus, it strongly resembles analogous pairings, studied in some of the references previously mentioned (cf. [3], [12], [25]). We prove that various methods of constructing a pairing on Hochschild homology lead to the same result. Notice that in [23], A. Ramadoss studied the links between different pairing on Hochschild homology.

To conclude, we would like to mention the recent paper by A. Polishchuk and A. Vaintrob [21] where a categorical version of the Riemann-Roch theorem was applied in the setting of the so-called Landau-Ginzburg models (the categories of matrix factorizations). We hope that our results, in combination with some results by D. Murfet [18], may provide an alternative way to derive the Riemann-Roch formula for singularities.

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2. Conventions

All along this paper k is a field of characteristic zero. A k-algebra is a k-module A equiped with an associative k-linear multiplication admitting a two sided unit 1_A.

All the graded modules considered in this paper are cohomologically $\mathbb{Z}$-graded. We abbreviate differential graded module (resp. algebra) by dg module (resp. dg algebra).

If A is a dg algebra and M and N are dg A-modules, we write $\mathrm{Hom}^\bullet_A(M,N)$ for the total Hom-complex.

If M is a dg k-module, we define $M^* = \mathrm{Hom}^\bullet_k(M,k)$ where k is considered as the dg k-module whose 0-th components is k and other components are zero.

We write $\otimes$ for the tensor product over k. If x is an homogeneous element of a differential graded module we denote by $|x|$ its degree.

If A is a dg algebra we will denote by A^{op} the opposite dg algebra. It is the same as a differential graded k-module but the multiplication is given by $a \cdot b = (-1)^{|a||b|}ba$. We denote by A^e the dg algebra $A \otimes A^{\mathrm{op}}$ and by ${}^eA$ the algebra $A^{\mathrm{op}} \otimes A$.

By a module we understand a left module unless otherwise specified. If A and B are dg algebras, A-B bimodules will be considered as left $A \otimes B^{\mathrm{op}}$-modules via the action

$$a \otimes b \cdot m = (-1)^{|b||m|}amb.$$

If we want to emphasize the left (resp. right) structure of an A-B bimodule we will write ${}_AM$ (resp. M_B). If M is an $A \otimes B^{\mathrm{op}}$-modules, then we write M^{op} for the corresponding $B^{\mathrm{op}} \otimes A$-module. Notice that $(M^{\mathrm{op}})^* \simeq (M^*)^{\mathrm{op}}$ as $B \otimes A^{\mathrm{op}}$-modules.

3. Perfect modules

3.1. Compact objects. — We recall some classical facts concerning compact objects in triangulated categories. We refer the reader to [20].

Let $\mathscr{T}$ be a triangulated category admitting arbitrary small coproducts.

Definition 3.1. — An object M of $\mathscr{T}$ is compact if for each family $(M_i)_{i\in I}$ of objects of $\mathscr{T}$ the canonical morphism

$$\bigoplus_{i\in I} \mathrm{Hom}_{\mathscr{T}}(M, M_i) \to \mathrm{Hom}_{\mathscr{T}}(M, \bigoplus_{i\in I} M_i) \tag{3.1}$$

is an isomorphism. We denote by $\mathscr{T}^c$ the full subcategory of $\mathscr{T}$ whose objects are the compact objects of $\mathscr{T}$.

Recall that a triangulated subcategory of $\mathscr{T}$ is called thick if it is closed under isomorphisms and direct summands.

Proposition 3.2. — *The category $\mathscr{T}^c$ is a thick subcategory of $\mathscr{T}$.*

We prove the following fact that will be of constant use.