Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 141
Fascicule 2

2013

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 299-342

Bull. Soc. math. France
141 (2), 2013, p. 299–342

SPECIALIZATION TO THE TANGENT CONE AND WHITNEY EQUISINGULARITY

by Arturo Giles Flores

Abstract. — Let $(X, 0)$ be a reduced, equidimensional germ of an analytic singularity with reduced tangent cone $(C_{X,0}, 0)$. We prove that the absence of exceptional cones is a necessary and sufficient condition for the smooth part $\mathfrak{X}^0$ of the specialization to the tangent cone $\varphi : \mathfrak{X} \to \mathbb{C}$ to satisfy Whitney's conditions along the parameter axis Y. This result is a first step in generalizing to higher dimensions Lê and Teissier's result for hypersurfaces of $\mathbb{C}^3$ which establishes the Whitney equisingularity of X and its tangent cone under these conditions.

Résumé (*Spécialisation sur le cône tangent et équisingularité à la Whitney*)
Soit $(X, 0)$ un germe de singularité analytique complexe, réduit et équidimensionel tel que son cône tangent $(C_{X,0}, 0)$ est réduit. On montre que l'absence des cônes exceptionnels est une condition nécessaire et suffisante pour que la partie lisse $\mathfrak{X}^0$ de la spécialisation sur le cône tangent $\varphi : \mathfrak{X} \to \mathbb{C}$ satisfasse les conditions de Whitney le long l'axe des paramètres Y. Ce résultat est un premier pas vers la généralisation aux dimensions supérieures du résultat de Lê et Teissier pour les hypersurfaces de $\mathbb{C}^3$ qui établit la équisingularité à la Whitney de X et son cône tangent sous ces conditions.

Texte reçu le 17 février 2011, révisé le 3 septembre 2011 et accepté le 16 mars 2012.

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2010 Mathematics Subject Classification. — 14J17, 32S15.

Key words and phrases. — Equisingularity, Whitney conditions, Specialization to the tangent cone.

It is my pleasure to thank Bernard Teissier who patiently taught me everything I know about this subject.

1. Introduction

The goal of this paper is to take a step in the study of the geometry of the specialization space $\varphi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$ of a germ of reduced and d-dimensional singularity $(X, 0)$ to its tangent cone $C_{X,0}$ from the point of view of Whitney equisingularity. The map φ describes a flat family of analytic germs with a section $\mathfrak{X} \overset{\curvearrowleft}{\to} \mathbb{C} : \sigma$, such that for each $t \in \mathbb{C}^*$ the germ $(\varphi^{-1}(t), \sigma(t))$ is isomorphic to $(X, 0)$ and the special fiber is isomorphic to the tangent cone. This construction is essentially due to Gerstenhaber [4] in a more algebraic setting.

One would like to establish conditions on the strata of the canonical Whitney stratification of a reduced complex analytic germ which ensure the Whitney equisingularity of the germ and its tangent cone. In this paper we achieve the "codimension zero" part of this program.

The space $(\mathfrak{X}, 0) \to (\mathbb{C}, 0)$ has been used to study Whitney conditions in [10], and to study the structure of the set of limits of tangent spaces in [16] and [15]. In [16], the authors prove the existence of a finite family $\{V_\alpha\}$ of subcones of the reduced tangent cone $|C_{X,0}|$ that determines the set of limits of tangent spaces to X at 0.

To be more specific, we fix an embedding $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ and build the normal/conormal diagram,

$$\begin{array}{ccc} E_0C(X) & \xrightarrow{\hat{e}_0} & C(X) \\ \downarrow{\scriptstyle \kappa'} & \overset{\xi}{\searrow} & \downarrow{\scriptstyle \kappa} \\ E_0X & \xrightarrow[e_0]{} & X \end{array}$$

where $E_0X \subset X \times \mathbb{P}^n$ is the blowup of X at the origin, $C(X) \subset X \times \check{\mathbb{P}}^n$ is the conormal space of X whose fiber determines the set of limits of tangent spaces (see section 4), and $E_0C(X) \subset X \times \mathbb{P}^n \times \check{\mathbb{P}}^n$ is the blowup in $C(X)$ of the subspace $\kappa^{-1}(0)$; consider the irreducible decomposition of the reduced fiber $|\xi^{-1}(0)| = \bigcup D_\alpha$. The authors prove that the fiber $\xi^{-1}(0)$ is contained in the incidence variety $I \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ and that each D_α establishes a projective duality of its images $V_\alpha \subset \mathbb{P}C_{X,0} \subset \mathbb{P}^n$ and $W_\alpha \subset \kappa^{-1}(0) \subset \check{\mathbb{P}}^n$.

In particular, the V_α's that are not irreducible components of the tangent cone are called exceptional cones and they appear in $\mathfrak{X}$ as an obstruction to the a_f stratification of the morphism $\mathfrak{X} \to \mathbb{C}$. They also prove that if the germ $(X, 0)$ is a cone itself, then it does not have exceptional cones. So a natural question arises, if a germ of an analytic singularity $(X, 0)$ does not have exceptional tangents, how close is it to being a cone?

A partial answer to this question was given in [15] in terms of Whitney equisingularity. The authors prove that for a surface $(S,0) \subset (\mathbb{C}^3,0)$ with reduced tangent cone $C_{S,0}$, the absence of exceptional cones is a necessary and sufficient condition for it to be Whitney equisingular to its tangent cone.

The specialization space $(\mathfrak{X},0) \to (\mathbb{C},0)$ has a canonical section which picks the origin in each fiber (see section 2). Let $Y \subset \mathfrak{X}$ be given by this section and let $\mathfrak{X}^0$ be the non-singular part $\mathfrak{X}$. The main objective of this paper is to prove that if the germ $(X,0)$ does not have exceptional cones and the tangent cone is reduced, then the couple $(\mathfrak{X}^0, Y)$ satisfies Whitney's conditions a) and b) at the origin.

2. Specialization to the tangent cone.

Let $(X,0)$ be a reduced germ of an analytic singularity of pure dimension d, with tangent cone $C_{X,0}$. Recall that the projectivized tangent cone can be defined as the exceptional divisor of the blowup of X in 0, and it is equivalent to considering the analytic "proj" of the graded algebra

$$gr_{\mathfrak{m}} O_{X,0} \colon = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

where $\mathfrak{m}$ is the maximal ideal of the analytic algebra $O_{X,0}$ associated to the germ. Moreover, if we consider an embedding $(X,0) \subset (\mathbb{C}^{n+1},0)$, the analytic algebra $O_{X,0}$ is isomorphic to $\mathbb{C}\{z_0,\dots,z_n\}/I$, where I is an ideal, $gr_{\mathfrak{m}} O_{X,0}$ is isomorphic to $\mathbb{C}[z_0,\dots,z_n]/\mathrm{In}_{\mathfrak{M}} I$ where $\mathfrak{M}$ is the maximal ideal of $\mathbb{C}\{z_0,\dots,z_n\}$, and the ideal $\mathrm{In}_{\mathfrak{M}} I$ is generated by all the initial forms with respect to the $\mathfrak{M}$-adic filtration of elements of I.

Let us suppose that the generators $\langle f_1,\dots,f_p\rangle$ for I, were chosen in such a way that their initial forms generate the ideal $\mathrm{In}_{\mathfrak{M}} I$ defining the tangent cone. Note that the f_i's are convergent power series in $\mathbb{C}^{n+1}$, so if m_i denotes the degree of the initial form of f_i, by defining

$$F_i(z_0,\dots,z_n,t) := t^{-m_i} f_i(tz_0,\dots,tz_n) \qquad (1)$$

we obtain convergent power series, defining holomorphic functions on a suitable open subset U of $\mathbb{C}^{n+1} \times \mathbb{C}$. Moreover, we can define the analytic algebra

$$O_{\mathfrak{X},0} = \mathbb{C}\{z_0,\dots,z_n,t\}/\langle F_1,\dots,F_p\rangle$$

with a canonical morphism $\mathbb{C}\{t\} \longrightarrow O_{\mathfrak{X},0}$ coming from the inclusion $\mathbb{C}\{t\} \hookrightarrow \mathbb{C}\{z_0,\dots,z_n,t\}$. Corresponding to this morphism of analytic algebras, we have the map germ $\varphi : (\mathfrak{X},0) \to (\mathbb{C},0)$ induced by the projection of $\mathbb{C}^{n+1} \times \mathbb{C}$ to the second factor.

Definition 2.1. — *The germ of analytic space over* $\mathbb{C}$,

$$\varphi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$$

is called the specialization of $(X, 0)$ *to its tangent cone* $(C_{X,0}, 0)$.

There is another way of building this space that will allow us to derive some interesting properties. Let $E_{(0,0)}\mathbb{C}^{n+2}$ be the blowup of the origin of $\mathbb{C}^{n+2}$, where we now have the coordinate system $(z_0, \dots, z_n, t)$. Let $W \subset E_{(0,0)}\mathbb{C}^{n+2}$ be the chart where the invertible ideal defining the exceptional divisor is generated by t, that is, in this chart the blowup map is given by $(z_0, \dots, z_n, t) \mapsto (tz_0, \dots, tz_n, t)$.

$$\begin{array}{ccc} W & \hookrightarrow & E_{(0,0)}\mathbb{C}^{n+2} \\ & \searrow & \downarrow E_0 \\ & & \mathbb{C}^{n+2} \end{array}$$

Lemma 2.2. — *Let* $X \times \mathbb{C} \subset \mathbb{C}^{n+2}$ *be a small enough representative of the germ* $(X \times \mathbb{C}, 0)$. *If* $(X \times \mathbb{C})'$ *denotes the strict transform of* $(X \times \mathbb{C})$ *in the blowup* $E_{(0,0)}\mathbb{C}^{n+2}$, *then the space* $(X \times \mathbb{C})' \bigcap W$ *together with the map induced by the restriction of the map* $E_{(0,0)}\mathbb{C}^{n+2} \to \mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}$ *is isomorphic to the specialization space* $\varphi : \mathfrak{X} \to \mathbb{C}$.

Proof. — We know that the strict transform $(X \times \mathbb{C})'$ is isomorphic to the blowup of $X \times \mathbb{C}$ at the origin, and we are seeing it as a reduced analytic subvariety of $\mathbb{C}^{n+2} \times \mathbb{P}^{n+1}$. This means that the exceptional divisor $(X \times \mathbb{C})' \cap (\{0\} \times \mathbb{P}^{n+1})$ is equal to $\mathbb{P}(C_{X,0} \times \mathbb{C})$, and so the ideal defining it is generated by the ideal defining the tangent cone $C_{X,0}$ in $\mathbb{C}^{n+1}$, that is, the ideal of initial forms $\mathrm{In}_{\mathfrak{M}} I$. By hypothesis, $W \subset E_{(0,0)}\mathbb{C}^{n+2} \subset \mathbb{C}^{n+2} \times \mathbb{P}^{n+1}$ is set theoretically described by

$$W = \big\{(tz_0, \dots, tz_n, t), [z_0 : \cdots : z_n : 1] \,|\, (z_0, \dots, z_n, t) \in \mathbb{C}^{n+2}\big\}$$

so in local coordinates the map E_0 restricted to W is given by $(z_0, \dots, z_n, t) \mapsto (tz_0, \dots, tz_n, t)$. Finally, since the ideal defining $X \times \mathbb{C}$ is generated in $\mathbb{C}\{z_0, \dots, z_n, t\}$ by the ideal $I = \langle f_1, \dots, f_p \rangle$ of $\mathbb{C}\{z_0, \dots, z_n\}$ defining X in $\mathbb{C}^{n+1}$, and since we have chosen the f_i's in such a way that their initial forms generate the ideal $\mathrm{In}_{\mathfrak{M}} I$, then the ideal defining the strict transform $(X \times \mathbb{C})'$ in W is given by

$$\mathfrak{J} O_W = \big\langle t^{-m_1} f_1(tz_0, \dots, tz_n), \dots, t^{-m_p} f_p(tz_0, \dots, tz_n) \big\rangle \, O_W$$

that is, we find the same functions $F_1, \dots, F_p$ which we used to define $\varphi : \mathfrak{X} \to \mathbb{C}$. □