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# COMPACT KÄHLER MANIFOLDS WITH COMPACTIFIABLE UNIVERSAL COVER

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Abstract. — Let $X$ be a compact Kähler manifold such that the universal cover admits a compactification. We conjecture that the fundamental group is almost abelian and reduce this problem to a classical conjecture of Iitaka.

Résumé (*Variétés kähleriennes compactes à revêtement universel compactifiable*)
Nous étudions les variétés kählériennes compactes dont le revêtement universel se réalise comme un ouvert de Zariski d'une variété compacte. Nous formulons la conjecture selon laquelle le groupe fondamental d'une telle variété devrait être virtuellement abélien et nous ramenons ce problème à une conjecture classique d'Iitaka.

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## 1. Introduction

The aim of this paper is to study the following problem.

1.1. — **Conjecture.** *Let $X$ be a compact Kähler manifold with infinite fundamental group $\pi_1(X)$. Suppose that the universal cover $\tilde{X}_{\mathrm{univ}}$ is a Zariski open subset $\tilde{X}_{\mathrm{univ}} \subset \overline{X}$ of some compact complex manifold $\overline{X}$. Then (after finite étale cover) there exists a locally trivial fibration $X \to A$ with simply connected fibre $F$ onto a complex torus $A$. In particular we have $\tilde{X}_{\mathrm{univ}} \simeq F \times \mathbb{C}^{\dim A}$.*

This conjecture generalises Iitaka's classical conjecture claiming that a compact Kähler manifold $X$ uniformised by $\mathbb{C}^{\dim X}$ is an étale quotient of a complex torus. In a recent paper with J. Kollár we studied this conjecture in the algebraic setting, i.e. under the additional hypothesis that $X$ is projective and $\tilde{X}_{\mathrm{univ}}$ is quasi-projective. It turned out that the key issue is to show that the fundamental group is almost abelian and we established the following statement.

1.2. — **Proposition.** [14, Prop.1.3] *Let $X$ have the smallest dimension among all normal, projective varieties that have an infinite, quasi-projective, étale Galois cover $\tilde{X} \to X$ whose Galois group is not almost abelian.*

*Then $X$ is smooth and its canonical bundle $K_X$ is nef but not semiample. (That is, $(K_X \cdot C) \geq 0$ for every algebraic curve $C \subset X$ but $\mathcal{O}_X(mK_X)$ is not generated by global sections for any $m > 0$.)*

By the *abundance conjecture* [31, Sec.2] the canonical bundle should always be semiample if it is nef. We then proved that in the algebraic case Conjecture 1.1 is implied by the abundance conjecture [14, Thm.1.1].

Since an infinite cover $\tilde{X} \to X$ is never an algebraic morphism, it is natural to look for an analogue of Proposition 1.2 in the analytic category. Note first that it is natural to impose that $X$ is Kähler: as we know from Hodge theory the Kähler condition establishes a link between the complex and the differentiable (i.e. topological) structure of $X$. Moreover there exist plenty of non-Kähler compact manifolds covered by compactifiable complex spaces, the easiest examples being Hopf manifolds [14, 1.6]. Although the existence of a compactification $\tilde{X} \subset \overline{X}$ should already be quite restrictive we will see that the appropriate analytic analogue of the quasiprojectiveness is the existence of a Kähler compactification.

1.3. — **Theorem.** *Let $X$ have the smallest dimension among all normal, compact Kähler spaces that have an infinite, étale Galois cover $\tilde{X} \to X$ whose*

*Galois group* $\Gamma$ *is not almost abelian and such that there exists a Kähler compactification* $\tilde{X} \subset \overline{X}$. *Then* $X$ *is smooth, does not admit any Mori contraction*[(1)], *and* $\tilde{X}$ *is not covered by positive-dimensional compact subspaces.*

*In particular* $X$ *has* $\pi_1$*-general type*[(2)], *i.e.* $\tilde{X}_{\mathrm{univ}}$ *is not covered by positive-dimensional compact subspaces.*

Even in the algebraic case, this statement gives some new information: if $X$ is projective, the absence of Mori contractions implies that $K_X$ is nef. Thus the "minimal dimensional counterexample" in Proposition 1.2 is of $\pi_1$-general type. Note also that for a manifold of $\pi_1$-general type the Conjecture 1.1 simply claims that $X$ is an étale quotient of a torus. Thus we are reduced to Iitaka's conjecture which has been studied by several authors [30, 29, 11, 20][(3)].

An important difference between the proof of Theorem 1.3 and the arguments in [14] is that the natural maps attached to compact Kähler manifolds (algebraic reduction, reduction maps for covering families of algebraic cycles) are in general not morphisms, as opposed to the classification theory of projective manifolds where we have Mori contractions and, assuming abundance, the Iitaka fibration at our disposal. Our key observation will be that for a general fibre of the $\Gamma$-reduction $\gamma$ (cf. Definition 2.2) the aforementioned meromorphic maps are holomorphic. We then deduce a strong dichotomy: up to replacing $\gamma$ by some factorisation the general fibre $G$ is either projective or does not contain any positive-dimensional compact proper subspaces (cf. Theorem 2.13). In a similar spirit F. Campana shows in the Appendix (Theorem A.8) from a more general viewpoint that Iitaka's conjecture has only to be treated for projective manifolds and simple compact Kähler manifolds, i.e. those which are not covered by positive-dimensional compact proper subspaces.

If we try to avoid the Kähler assumption on $\overline{X}$ we still obtain some information of bimeromorphic nature:

1.4. — **Proposition.** *Let* $X$ *have the smallest dimension among all normal, compact Kähler spaces that have an infinite, étale Galois cover* $\tilde{X} \to X$ *whose Galois group* $\Gamma$ *is not almost abelian and such that there exists a compactification* $\tilde{X} \subset \overline{X}$. *Then* $X$ *is smooth and special in the sense of Campana* [9].

(1) In the analytic setting we define a Mori contraction as a proper holomorphic morphism with connected fibres $\mu : X \to X'$ onto a normal complex space $X'$ such that $-K_X$ is $\mu$-ample.
(2) We follow the terminology of [7], this corresponds to the property of $X$ having a generically large fundamental group in the sense of [24].
(3) Apart from [11] these papers do not really use that $\tilde{X}_{\mathrm{univ}} \simeq \mathbb{C}^{\dim X}$.

This proposition follows rather quickly from an orbifold version of the Kobayashi-Ochiai theorem (Theorem 3.1). By results of F. Campana and the first named author [9, Thm.3.33], [10, Thm.1.1] the fundamental group of a special manifold of dimension at most three is almost abelian, so our counterexample (if it exists) would have $\dim X \geq 4$.

Let us finally note that once we have understood the fundamental group, the geometric statement in Conjecture 1.1 is not far away.

1.5. — **Theorem.** *Let $X$ be a compact Kähler manifold whose universal cover $\tilde{X}_{\mathrm{univ}}$ admits a Kähler compactification $\tilde{X}_{\mathrm{univ}} \subset \overline{X}$. If the fundamental group of $X$ is almost abelian, the Albanese map of $X$ is (up to finite étale cover) a locally trivial fibration whose fibre $F$ is simply connected.*

Since the proof of the corresponding statement in the algebraic setting [14, Thm.1.4] relies on strong results of Hodge theory for birational morphisms which are unknown in the Kähler setting, our argument follows the lines of [25]. Indeed if $X$ is of $\pi_1$-general type, [25, Thm. 16] implies that $X$ is isomorphic to its Albanese torus (even without any further assumption on $\overline{X}$); see [16] and Remark 3.3 for a discussion around this general case.

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## 2. Notation and basic results

Manifolds and complex spaces will always be supposed to be irreducible.

If $X$ is a normal complex space we denote by $\mathscr{C}(X)$ its cycle space [1]. We will use very often that if $X$ is a compact Kähler space, then the irreducible components of $\mathscr{C}(X)$ are compact (Bishop's theorem, see [26]).

A fibration is a proper surjective map $\varphi : X \to Y$ with connected fibres between normal complex varieties. A meromorphic map $\varphi : X \to Y$ is almost holomorphic if there exists a Zariski open dense subset $X^0 \subset X$ such that the restriction $\varphi|_{X^0}$ is holomorphic and $\varphi|_{X^0} : X^0 \to Y$ is a proper map.

Recall that a fibration $\varphi : X \to Y$ from a manifold $X$ onto a normal complex space $Y$ is almost smooth if the reduction $F_{\mathrm{red}}$ of every fibre is smooth and has the expected dimension. In this case the complex space $Y$ has at most quotient singularities, the local structure around $y \in Y$ being given by a finite representation of the fundamental group of $\pi_1(F_{\mathrm{red}})$ [28, Prop.3.7]. Thus there exists *locally* a finite base change $Y' \to Y$ such that the normalisation $X'$ of $X \times_Y Y'$ is smooth over $Y'$.