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STRONG ALMOST REDUCIBILITY FOR ANALYTIC AND GEVREY QUASI-PERIODIC COCYCLES

by Claire Chavaudret

Abstract. — This article is about almost reducibility of quasi-periodic cocycles with a diophantine frequency which are sufficiently close to a constant. Generalizing previous works by L.H. Eliasson, we show a strong version of almost reducibility for analytic and Gevrey cocycles, that is to say, almost reducibility where the change of variables is in an analytic or Gevrey class which is independent of how close to a constant the initial cocycle is conjugated. This implies a result of density, or quasi-density, of reducible cocycles near a constant. Some algebraic structure can also be preserved, by doubling the period if needed.

Résumé (*Presque réductibilité forte pour les cocycles quasi-périodiques de classe analytique et Gevrey*)

Cet article traite de la presque-réductibillité des cocycles quasi-périodiques à fréquence diophantienne qui sont proches d'un cocycle constant. Nous démontrons un résultat de presque-réductibilité forte des cocycles analytiques et Gevrey, c'est-à-dire que le changement de variables obtenu pour conjuguer le cocycle initial à un cocycle proche d'une constante est dans une classe analytique ou Gevrey qui est indépendante de la proximité à la constante; ceci généralise certains résultats antérieurs de L.H. Eliasson. Ce résultat a pour corollaire un théorème de densité ou de quasi-densité des cocycles réductibles au voisinage d'une constante. Il est possible de préserver certaines caractéristiques algébriques du cocycle initial en doublant la période.

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1. Introduction

We are concerned with quasi-periodic cocycles, that is, solutions of equations of the form

$$\forall(\theta,t)\in 2\mathbb{T}^d\times\mathbb{R},\ \frac{d}{dt}X^t(\theta)=A(\theta+t\omega)X^t(\theta);\ X^0(\theta)=Id \tag{1}$$

where $A\in C^0(2\mathbb{T}^d,\mathcal{G})$ and $\mathcal{G}$ is a linear Lie algebra. Here $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$ stands for the d-torus, $d\geq 1$, and $2\mathbb{T}^d=\mathbb{R}^d/(2\mathbb{Z}^d)$ stands for the double torus. We will assume in this article that $\omega\in\mathbb{R}^d$ satisfies some diophantine conditions. The solution of (1) is called the quasi-periodic cocycle associated to A and is defined on $2\mathbb{T}^d\times\mathbb{R}$ with values in the connected component of the identity of a Lie group G whose associated Lie algebra is $\mathcal{G}$. Terminology is explained by the fact that A is the envelope of a quasi-periodic function, since $t\mapsto A(\theta+t\omega)$ is a quasi-periodic function for all $\theta\in 2\mathbb{T}^d$. We say X is a constant cocycle if A is constant. A constant cocycle is always of the form $t\mapsto e^{tA}$.

A cocycle is said to be reducible if it is conjugated to a constant cocycle, in a sense that will be defined later on. The problem of reducibility of cocycles has been thoroughly studied and is of interest because the dynamics of reducible cocycles is well understood and because this problem has links with the spectral theory of Schrödinger cocycles and with the problem of lower dimensional invariant tori in hamiltonian systems. In the periodic case ($d=1$), Floquet theory tells that every cocycle is reducible modulo a loss of periodicity. However, the problem is far more difficult if d is greater than 1 and it is not true that every cocycle is then reducible. The question becomes whether every cocycle is close, up to a conjugacy, to a reducible one; from this question comes the notion of almost-reducibility.

For any functional class $\mathcal{C}$, a cocycle is said to be *almost-reducible in* $\mathcal{C}$ if it can be conjugated to a cocycle which is arbitrarily close in the topology of $\mathcal{C}$ to a reducible one, with the conjugacy also in $\mathcal{C}$. Reductibility implies almost reducibility, however the reverse is not true: there are non reducible cocycles even close to a constant cocycle (see [3]). Almost reducibility is an interesting notion since the dynamics of an almost reducible cocycle are quite well known on a very long time.

We first focus on cocycles generated by functions which are analytic on a neighbourhood of the torus, i.e real analytic functions which are periodic in the direction of the real axis (recall that they are matrix-valued). For such a function F, we will let

$$|F|_r=\sup_{|\mathrm{Im}\,\theta|\leq r}||F(\theta)||$$

where $||\,.\,||$ stands for the operator norm.

The aim of this paper is to show that for

$$G = GL(n,\mathbb{C}), GL(n,\mathbb{R}), SL(2,\mathbb{C}), SL(n,\mathbb{R}), Sp(n,\mathbb{R})^{(1)}, O(n), U(n)$$

in the neighbourhood of a constant cocycle, i.e under a smallness condition on the non-constant part of the cocycle, every cocycle which is analytic on an r-neighbourhood of the torus and G-valued is almost reducible in $C^\omega_{r'}(2\mathbb{T}^d, G)$ for all $0 < r' < r \leq \frac{1}{2}$, in the sense defined above. The smallness condition only depends on the dimensions n, d, on the diophantine class of ω, on the constant cocycle and on the loss of analyticity $r - r'$.

More precisely, we shall prove the following theorem, for G among the groups cited above and $\mathcal{G}$ the Lie algebra associated to G:

Theorem 1.1. — *Let $0 < r' < r \leq \frac{1}{2}$, $A \in \mathcal{G}$, $F \in C^\omega_r(\mathbb{T}^d, \mathcal{G})$. There is $\epsilon_0 < 1$ depending only on $n, d, \omega, A, r - r'$ such that if*

$$|F|_r \leq \epsilon_0$$

then for all $\epsilon > 0$, there exists $\bar{A}_\epsilon, \bar{F}_\epsilon \in C^\omega_{r'}(2\mathbb{T}^d, \mathcal{G})$, $\Psi_\epsilon, Z_\epsilon \in C^\omega_{r'}(2\mathbb{T}^d, G)$ and $A_\epsilon \in \mathcal{G}$ such that for all $\theta \in 2\mathbb{T}^d$,

$$\partial_\omega Z_\epsilon(\theta) = (A + F(\theta))Z_\epsilon(\theta) - Z_\epsilon(\theta)(\bar{A}_\epsilon(\theta) + \bar{F}_\epsilon(\theta))$$

with

1. $\partial_\omega \Psi_\epsilon = \bar{A}_\epsilon \Psi_\epsilon - \Psi_\epsilon A_\epsilon$,
2. $|\bar{F}_\epsilon|_{r'} \leq \epsilon$,
3. $|\Psi_\epsilon|_{r'} \leq \epsilon^{-\frac{1}{8}}$,
4. *and* $|Z_\epsilon - Id|_{r'} \leq 2\epsilon_0^{\frac{1}{2}}$.

Moreover, if $G \subset GL(2,\mathbb{C})$ or if $G = GL(n,\mathbb{C})$ or $U(n)$, $Z_\epsilon, \bar{A}_\epsilon, \bar{F}_\epsilon$ are in $C^\omega_{r'}(\mathbb{T}^d)$.

Property 1 states the reducibility of $\bar{A}_\epsilon$. Theorem 1.1 immediately entails the following:

Theorem 1.2. — *Let $0 < r' < r \leq \frac{1}{2}$, $A \in \mathcal{G}$, $F \in C^\omega_r(\mathbb{T}^d, \mathcal{G})$. There is $\epsilon_0 < 1$ depending only on $n, d, \omega, A, r - r'$ such that if*

$$|F|_r \leq \epsilon_0$$

then for all $\epsilon > 0$, there exists $F_\epsilon \in C^\omega_{r'}(2\mathbb{T}^d, \mathcal{G})$, $Z_\epsilon \in C^\omega_{r'}(2\mathbb{T}^d, G)$ and $A_\epsilon \in \mathcal{G}$ such that for all $\theta \in 2\mathbb{T}^d$,

$$\partial_\omega Z_\epsilon(\theta) = (A + F(\theta))Z_\epsilon(\theta) - Z_\epsilon(\theta)(A_\epsilon + F_\epsilon(\theta))$$

with $|F_\epsilon|_{r'} \leq \epsilon$.

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(1) With n even.

Note that in Theorem 1.2, we do not have any good estimate of Z_ϵ. Theorem 1.1 also holds if one chooses F in a class which is bigger than $C_r^\omega(\mathbb{T}^d,\mathcal{G})$, i.e the class of functions in $C_r^\omega(2\mathbb{T}^d,\mathcal{G})$ satisfying some "nice periodicity properties" with respect to the matrix A.

There is a loss of analyticity in this result, but it is arbitrarily small. A result close to Theorem 1.1 in the case when $G=GL(n,\mathbb{R})$ had already been proven in [5] by L.H. Eliasson:

THEOREM [ELIASSON] . — *Let $A\in gl(n,\mathbb{R})$ and $F\in C_r^\omega(\mathbb{T}^d,gl(n,\mathbb{R}))$. There is $\epsilon_0<1$ depending only on $n,d,\kappa,\tau,||A||,r$ such that if $|F|_r\leq\epsilon_0$, then for all $\epsilon>0$, there exists $0<r_\epsilon<r$, $Z_\epsilon\in C_{r_\epsilon}^\omega(2\mathbb{T}^d,GL(n,\mathbb{R}))$ such that for all $\theta\in 2\mathbb{T}^d$,*

$$\partial_\omega Z_\epsilon(\theta)=(A+F(\theta))Z_\epsilon(\theta)-Z_\epsilon(\theta)(A_\epsilon+F_\epsilon(\theta))$$

with $A_\epsilon\in gl(n,\mathbb{R})$, $F_\epsilon\in C_{r_\epsilon}^\omega(2\mathbb{T}^d,gl(n,\mathbb{R}))$ and $|F_\epsilon|_{r_\epsilon}\leq\epsilon$.

Eliasson's theorem merely states almost reducibility in $\cup_{r'>0}C_{r'}^\omega(2\mathbb{T}^d,GL(n,\mathbb{R}))$, since the sequence (r_ϵ) might well tend to 0. The achievement of Theorem 1.1 is to state almost reducibility in a more general algebraic framework, but also, and mostly, to show that almost reducibility holds in a *fixed neighbourhood* of a torus even when this torus has dimension greater than 1. This is almost reducibility in a strong sense.

Note that, as was the case in [5], one cannot avoid to lose periodicity in Theorem 1.1 if G is a real group with dimension greater than 2. The notion of "nice periodicity properties" that will be given aims at limiting this loss to a period doubling. In comparison with the real framework, the symplectic framework does not introduce any new constraints in the elimination of resonances (Section 2.2); therefore there is no more loss of periodicity here than in the case when $G=GL(n,\mathbb{R})$. As before in [2], a single period doubling is sufficient in the case when G is a real symplectic group.

The second part of this paper is dedicated to showing that the same method gives an analogous result for cocycles which are in a Gevrey class (Theorem 3.1); denoting by $C_r^{G,\beta}$ the class of Gevrey functions with exponent β and parameter r (so that $C_r^{G,1}$ is the class of analytic functions), and by $||\ .\ ||_{\beta,r}$ their norm, we have the following:

THEOREM 1.3. — *Let $0<r'<r\leq\frac{1}{2}$, $A\in\mathcal{G}$, $F\in C_r^{G,\beta}(\mathbb{T}^d,\mathcal{G})$. There is $\epsilon_0<1$ depending only on $n,d,\kappa,\tau,A,r-r'$ such that if*

$$||F||_{\beta,r}\leq\epsilon_0$$

then for all $\epsilon>0$, there exists $\bar{A}_\epsilon,\bar{F}_\epsilon\in C_{r'}^{G,\beta}(2\mathbb{T}^d,\mathcal{G})$, $\Psi_\epsilon,Z_\epsilon\in C_{r'}^{G,\beta}(2\mathbb{T}^d,G)$ and $A_\epsilon\in\mathcal{G}$ such that for all $\theta\in 2\mathbb{T}^d$,

$$\partial_\omega Z_\epsilon(\theta)=(A+F(\theta))Z_\epsilon(\theta)-Z_\epsilon(\theta)(\bar{A}_\epsilon(\theta)+\bar{F}_\epsilon(\theta))$$