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EXTREMAL KÄHLER METRICS ON BLOW-UPS OF PARABOLIC RULED SURFACES

by Carl Tipler

Abstract. — New examples of extremal Kähler metrics are given on blow-ups of parabolic ruled surfaces. The method used is based on the gluing construction of Arezzo, Pacard and Singer [5]. This enables to endow ruled surfaces of the form $\mathbb{P}(\mathcal{O}\oplus L)$ with special parabolic structures such that the associated iterated blow-up admits an extremal metric of non-constant scalar curvature.

Résumé (*Métriques de Kähler extrémales sur les éclatements de surfaces réglées paraboliques*)

De nouveaux exemples de métriques de Kähler extrémales sont donnés sur des éclatements de surfaces réglées paraboliques. La technique utilisée est basée sur la méthode de recollement de Arezzo, Pacard et Singer [5]. Ceci permet de munir les surfaces réglées de la forme $\mathbb{P}(\mathcal{O}\oplus L)$ de structures paraboliques particulières telles que les éclatements itérés associés supportent des métriques extrémales à courbure scalaire non constante.

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1. Introduction

In this paper is adressed the problem of existence of extremal Kähler metrics on ruled surfaces. An extremal Kähler metric on a compact Kähler manifold M is a metric that minimizes the Calabi functional in a given Kähler class Ω:

$$\begin{array}{ccc} \{\omega \in \Omega^{1,1}(M,\mathbb{R}), d\omega = 0,\ \omega > 0 \,/[\omega] = \Omega\} & \rightarrow & \mathbb{R} \\ \omega & \mapsto & \int_M s(\omega)^2 \omega^n. \end{array}$$

Here, $s(\omega)$ stands for the scalar curvature of ω and n is the complex dimension of M. Constant scalar curvature metrics are examples of extremal metrics. If the manifold is polarized by an ample line bundle L the existence of such a metric in the class $c_1(L)$ is related to a notion of stability of the pair (M, L). More precisely, the works of Yau [30], Tian [28], Donaldson [10] and lastly Székelyhidi [25], led to the conjecture that a polarized manifold (M, L) admits an extremal Kähler metric in the Kähler class $c_1(M)$ if and only if it is relatively K-polystable. So far it has been proved that the existence of a constant scalar curvature Kähler metric implies K-stability [18] and the existence of an extremal metric implies relative K-polystability [24].

We will focus on the special case of complex ruled surfaces. First consider a geometrically ruled surface M. This is the total space of a fibration

$$\mathbb{P}(E) \rightarrow \Sigma$$

where E is a holomorphic bundle of rank 2 on a compact Riemann surface Σ. In that case, the existence of extremal metrics is related to the stability of the bundle E. A lot of work has been done in this direction, we refer to [2] for a survey on this topic.

Moreover, in this paper, Apostolov, Calderbank, Gauduchon and Tønnesen-Friedman prove that if the genus of Σ is greater than two, then M admits a metric of constant scalar curvature in some class if and only if E is polystable. Another result due to Tønnesen-Friedman [29] is that if the genus of Σ is greater than two, then there exists an extremal Kähler metric of non-constant scalar curvature on M if and only if $M = \mathbb{P}(\mathcal{O} \oplus L)$ with L a line bundle of positive degree (see also [26]). Note that in that case the bundle is unstable.

The above results admit partial counterparts in the case of parabolic ruled surfaces (see definition 1.0.1). In the papers [20] and [21], Rollin and Singer showed that the parabolic polystability of a parabolic ruled surface S implies the existence of a constant scalar curvature metric on an iterated blow-up of S encoded by the parabolic structure.

It is natural to ask for such a result in the extremal case. If there exists an extremal metric of non-constant scalar curvature on an iterated blow-up of a parabolic ruled surface, the existence of the extremal vector field implies that M is of the form $\mathbb{P}(\mathcal{O} \oplus L)$. Moreover, the marked points of the parabolic structure must lie on the zero or infinity section of the ruling. Inspired by the results mentioned above, we can ask if for every unstable parabolic structure on a minimal ruled surface of the form $M = \mathbb{P}(\mathcal{O} \oplus L)$, with marked points on the infinity section of the ruling, one can associate an iterated blow-up of M supporting an extremal Kähler metric of non-constant scalar curvature.

Arezzo, Pacard and Singer, and then Székelyhidi, proved that under some stability conditions, one can blow-up an extremal Kähler manifold and obtain an extremal Kähler metric on the blown-up manifold for sufficiently small metric on the exceptional divisor. This blow-up process enables to prove that many of the unstable parabolic structures give rise to extremal Kähler metrics of non-constant scalar curvature on the associated iterated blow-ups. A modification of their argument will enable to get more examples of extremal metrics on blow-ups encoded by unstable parabolic structures.

In order to state the result, we need some definitions about parabolic structures. Let Σ be a Riemann surface and $\check{M}$ a geometrically ruled surface, total space of a fibration

$$\pi : \mathbb{P}(E) \to \Sigma$$

with E a holomorphic bundle.

Definition 1.0.1. — A *parabolic structure* $\mathscr{P}$ on

$$\pi : \check{M} = P(E) \to \Sigma$$

is the data of s distinct points $(A_i)_{1 \leq i \leq s}$ on Σ and for each of these points the assignment of a point $B_i \in \pi^{-1}(A_i)$ with a weight $\alpha_i \in (0,1) \cap \mathbb{Q}$. A geometrically ruled surface endowed with a parabolic structure is called a *parabolic ruled surface.*

In the paper [20], to each parabolic ruled surface is associated an iterated blow-up

$$\Phi : Bl(\check{M}, \mathscr{P}) \to \check{M}.$$

We will describe the process to construct $Bl(\check{M}, \mathscr{P})$ in the case of a parabolic ruled surface whose parabolic structure consists of a single point, the general case being obtained operating the same way for each marked point. Let $\check{M} \to \Sigma$ be such a parabolic ruled surface with $A \in \Sigma$, marked point $Q \in F := \pi^{-1}(A)$ and weight $\alpha = \dfrac{p}{q}$, with p and q coprime integers, $0 < p < q$. Denote the expansions of $\frac{p}{q}$ and $\frac{q-p}{q}$ into continuous fractions by:

$$\frac{p}{q} = \cfrac{1}{e_1 - \cfrac{1}{e_2 - \ldots \cfrac{1}{e_k}}}$$

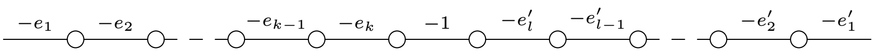
and

$$\frac{q-p}{q} = \cfrac{1}{e'_1 - \cfrac{1}{e'_2 - \ldots \cfrac{1}{e'_l}}}.$$

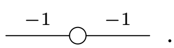
Suppose that the integers e_i and e'_i are greater or equal than two so that these expansions are unique. Then from [20] there exists a unique iterated blow-up

$$\Phi : Bl(\check{M}, \mathscr{P}) \to \check{M}$$

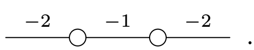
with $\Phi^{-1}(F)$ equal to the following chain of curves:



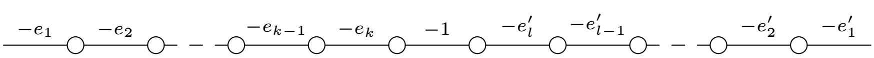
The edges stand for the rational curves, with self-intersection number above them. The dots are the intersection of the curves, of intersection number 1. Moreover, the curve of self-intersection $-e_1$ is the proper transform of the fiber F. In order to get this blow-up, start by blowing-up the marked point and obtain the following curves:



Here the curve on the left is the proper transform of the fiber and the one on the right is the first exceptional divisor. Then blow-up the intersection of these two curves to obtain



Then choosing one of the two intersection points that the last exceptional divisor gives and iterating the process, one obtain the following chain of curves



REMARK 1.0.2. — The chain of curves on the left of the one of self-intersection number -1 is the chain of a minimal resolution of a singularity of $A_{p,q}$ type and the one on the right of a singularity of $A_{q-p,q}$ type (see section 2).