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## EXCEPTIONALLY SMALL BALLS IN STABLE TREES

BY THOMAS DUQUESNE & GUANYING WANG

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**ABSTRACT.** — The  $\gamma$ -stable trees are random measured compact metric spaces that appear as the scaling limit of Galton-Watson trees whose offspring distribution lies in a  $\gamma$ -stable domain,  $\gamma \in (1, 2]$ . They form a specific class of Lévy trees (introduced by Le Gall and Le Jan in [24]) and the Brownian case  $\gamma = 2$  corresponds to Aldous Continuum Random Tree (CRT). In this paper, we study fine properties of the mass measure, that is the natural measure on  $\gamma$ -stable trees. We first discuss the minimum of the mass measure of balls with radius  $r$  and we show that this quantity is of order  $r^{\frac{\gamma}{\gamma-1}} (\log 1/r)^{-\frac{1}{\gamma-1}}$ . We think that no similar result holds true for the maximum of the mass measure of balls with radius  $r$ , except in the Brownian case: when  $\gamma = 2$ , we prove that this quantity is of order  $r^2 \log 1/r$ . In addition, we compute the exact constant for the lower local density of the mass measure (and the upper one for the CRT), which continues previous results from [9, 10, 13].

**RÉSUMÉ** (*Boules exceptionnellement petites dans les arbres stables*)

Les arbres  $\gamma$ -stables sont des espaces métriques compacts à mesure aléatoire qui apparaissent en tant que limite de mise à l'échelle d'arbres de Galton-Watson dont les distributions sont situées dans un domaine  $\gamma$ -stable,  $\gamma \in ]1, 2]$ . Ils forment une classe spécifique des arbres de Lévy (introduite par Le Gall et Le Jan dans [24]) et le cas brownien  $\gamma = 2$  correspond aux arbres aléatoires du continuum d'Aldous (CRT).

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Dans cet article nous étudions les propriétés fines de la mesure de masse, qui est la mesure naturelle des arbres  $\gamma$ -stables. Nous discutons d'abord le minimum de la mesure de masse des boules de rayon  $r$  et nous montrons que cette quantité est de l'ordre de  $r^{\frac{\gamma}{\gamma-1}} (\log 1/r)^{-\frac{1}{\gamma-1}}$ . Nous pensons qu'aucun résultat similaire n'est vrai pour le maximum des mesures de masse de boule de rayon  $r$ , sauf dans le cas brownien : quand  $\gamma = 2$  nous montrons que cette quantité est de l'ordre de  $r^2 \log 1/r$ . D'autre part, nous calculons la constante exacte de la densité local inférieure de la mesure de masse (et la supérieure pour le CRT), à la suite de résultats précédents de [9, 10, 13].

## 1. Introduction

Stable trees are particular instances of Lévy trees that form a class of random compact metric spaces introduced by Le Gall and Le Jan in [24] as the genealogy of Continuous State Branching Processes (CSBP for short). The class of stable trees contains Aldous's continuum random tree that corresponds to the Brownian case (see Aldous [2, 3]). Stable trees (and more generally Lévy trees) are the scaling limit of Galton-Watson trees (see [11] Chapter 2 and [8]). Various geometric and distributional properties of Lévy trees (and of stable trees, consequently) have been studied in [12] and in Weill [28]. Stable trees have been also studied in connection with fragmentation processes: see Miermont [25, 26], Haas and Miermont [20], Goldschmidt and Haas [18] for the stable cases and see Abraham and Delmas [1] for related models concerning more general Lévy trees. To study Brownian motion on stable trees, D. Croydon in [6] got partial results on balls with exceptionnally small mass measure.

Before stating the results, let us briefly explain the definition of stable trees before stating the main results of the paper. Let us fix the stable index  $\gamma \in (1, 2]$  and let  $X = (X_t)_{t \geq 0}$  be a spectrally positive  $\gamma$ -stable Lévy process that is defined on the probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . More precisely, we suppose that  $\mathbf{E}[\exp(-\lambda X_t)] = \exp(t\lambda^\gamma)$ ,  $t, \lambda \in [0, \infty)$ . Note that  $X$  is a Brownian motion when  $\gamma = 2$  and we shall refer to this case as to the *Brownian case*. As shown by Le Gall and Le Jan [24] (see also [11] Chapter 1), there exists a *continuous process*  $H = (H_t)_{t \geq 0}$  such that for any  $t \in [0, \infty)$ , the following limit holds true in probability

$$(1) \quad H_t := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{I_t^s < X_s < I_t^s + \varepsilon\}} ds.$$

Here  $I_t^s$  stands for  $\inf_{s \leq r \leq t} X_r$ . The process  $H$  is the  $\gamma$ -stable height process. Note that in the Brownian case,  $H$  is simply a reflected Brownian motion. Theorems 2.3.2 and 2.4.1 in [11] show that  $H$  is the scaling limit of the contour

function (or the depth-first exploration process) of an i.i.d. sequence of Galton-Watson trees whose offspring distribution is in the domain of attraction of a  $\gamma$ -stable law.

As in the discrete setting, the process  $H$  encodes a family of continuous trees: each excursion of  $H$  above 0 corresponds to the exploration process of a single continuous tree of the family. Let us make this statement more precise thanks to excursion theory. Recall that  $X$  has unbounded variation sample paths. We set  $I_t = \inf_{s \in [0, t]} X_s$ , that is the infimum process of  $X$ . Basic results on fluctuation theory (see Bertoin [4] VII.1) entail that  $X - I$  is a strong Markov process in  $[0, \infty)$  and that 0 is regular for  $(0, \infty)$  and recurrent with respect to this Markov process. Moreover,  $-I$  is a local time at 0 for  $X - I$  (see Bertoin [4] Theorem VII.1). We denote by  $\mathbf{N}$  the corresponding *excursion measure of  $X - I$  above 0*. We denote by  $(a_j, b_j)$ ,  $j \in \mathcal{J}$ , the excursion intervals of  $X - I$  above 0, and by  $X^j = X_{(a_j+, \cdot) \wedge b_j} - I_{a_j}$ ,  $j \in \mathcal{J}$ , the corresponding excursions. Next, observe that if  $t \in (a_j, b_j)$ , the value of  $H_t$  only depends on  $X^j$ . Moreover, one can show that  $\bigcup_{j \in \mathcal{J}} (a_j, b_j) = \{t \geq 0 : H_t > 0\}$ . This allows to define the height process under  $\mathbf{N}$  as a certain measurable function  $H(X)$  of  $X$ , that we simply denote by  $(H_t)_{t \geq 0}$ . For any  $j \in \mathcal{J}$ , we then set  $H^j = H_{(a_j+, \cdot) \wedge b_j}$  and the point measure

$$(2) \quad \sum_{j \in \mathcal{J}} \delta_{(-I_{a_j}, H^j)}$$

is distributed as a Poisson point measure on  $[0, \infty) \times C([0, \infty), \mathbb{R})$  with intensity  $\ell \otimes \mathbf{N}$ , where  $\ell$  stands for the Lebesgue measure on  $[0, \infty)$ . Note that  $X$  and  $H$  under  $\mathbf{N}$  are paths with the same lifetime given by

$$\zeta := \inf\{t \in (0, \infty) : H_t = 0\}.$$

Standard results in fluctuation theory imply that  $0 < \zeta < \infty$ ,  $\mathbf{N}$ -a.e. and that

$$\mathbf{N}(1 - e^{-\lambda\zeta}) = \lambda^{1/\gamma}, \quad \lambda \in [0, \infty).$$

Thus,  $\mathbf{N}(\zeta \in dr) = Cr^{-\frac{1}{\gamma}-1}\ell(dr)$ , where  $1/C = \gamma\Gamma(1 - \frac{1}{\gamma})$  (here,  $\Gamma$  stands for Euler's Gamma function). Note that  $\mathbf{N}$ -a.e.  $H_t > 0$  iff  $t \in (0, \zeta)$  and  $H_0 = H_t = 0$ , for any  $t \in [\zeta, \infty)$ . We refer to Chapter 1 in [11] for more details.

The excursion  $(H_t)_{0 \leq t \leq \zeta}$  under  $\mathbf{N}$  is the depth-first exploration process of a continuous tree that is defined as the following metric space: for any  $s, t \in [0, \zeta]$ , we set

$$(3) \quad b(s, t) = \min_{s \wedge t \leq r \leq s \vee t} H_r \quad \text{and} \quad d(s, t) = H_t + H_s - 2b(s, t).$$

The quantity  $b(s, t)$  is the height of the branching point between the vertices visited at times  $s$  and  $t$  and  $d(s, t)$  is therefore the distance in the tree of these vertices. We easily show that  $d$  is a pseudo-metric and we introduce the