Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# ON $p$-ADIC SPEH REPRESENTATIONS

Alexandru Ioan Badulescu

Tome 142
Fascicule 1

2014

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 255-267

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 1, tome 142, janvier 2014



***Diffusion***

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

***Tarifs***

*Vente au numéro :* 43 € ($ 64)
*Abonnement* Europe : 300 €, hors Europe : 334 € ($ 519)
Des conditions spéciales sont accordées aux membres de la SMF.

***Secrétariat : Nathalie Christiaën***

*Bulletin de la Société Mathématique de France*
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/



ISSN 0037-9484

Directeur de la publication : Marc Peigné

*Bull. Soc. math. France*
**142** (1), 2014, p. 255–267

# ON $p$-ADIC SPEH REPRESENTATIONS

BY ALEXANDRU IOAN BADULESCU

*À la Virgen del Rocío*

ABSTRACT. — This note contains simple proofs of some known results (unitarity, character formula) on Speh representations of a group $GL_n(D)$ where $D$ is a local non Archimedean division algebra of any characteristic.

RÉSUMÉ (*Sur les représentations de Speh p-adiques*). — Cette note contient des preuves simples de certains faits connus (unitarisabilité, formule des caractères) concernant les représentations de Speh d'un groupe $GL_n(D)$, où $D$ est une algèbre à division locale non-archimédienne de caractéristique quelconque.

## Introduction

In this note I give simple proofs of some known results on Speh representations of groups $GL_n(D)$ where $D$ is a central division algebra of finite dimension over a local non archimedean field $F$ of any characteristic. The new idea is to use the Moeglin-Waldspurger algorithm (MWA) for computing the dual of an

*Texte reçu le 23 décembre 2011, accepté le 9 mars 2012.*

ALEXANDRU IOAN BADULESCU, Université Montpellier 2, Département de mathématiques, case courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex, France • *E-mail :* badulesc@math.univ-poitiers.fr

2010 Mathematics Subject Classification. — 22D10, 11S37.

Key words and phrases. — Representations of $p$-adic groups, Langlands program, unitary representations.

irreducible representation. For the unitarizability of Speh representations, previous proofs were based either on the trace formula and the close fields theory, or on deep results of type theory. The proof here is "combinatoric", independent of $D$, the characteristic, and Bernstein's (also called U0) theorem. Short general proofs using MWA are also given for other known facts. The proof is always the same: one wants to prove a relation (R) involving some representations (for example an induced representation $\pi$ is irreducible). One starts by writing the "naive" relation (R$'$) between these representations, known from standard theory, but not as strong as (R) (for example the semi simplification of $\pi$ is a sum with non negative coefficients of some irreducible representations). Usually (R$'$) has more terms that (R), because it is weaker, and one wants to prove that some terms which are supposed to appear in (R$'$) are actually not there (for example all the subquotients of $\pi$ except of the expected one have coefficient zero). The method then is to consider also the dual relation (R$''$) to (R$'$), and to play with the MWA, in order to show that the mild constraints one has on (R$'$) and (R$''$) are enough to show that the extra terms are not there and (R$'$) reduces actually to (R).

All the results in this paper are already known, and here I only give new short proofs. So I kindly ask the reader, when using one of these facts, to quote, at least in first place, the original reference (see the historical notice at the end of the paper). Beside the Zelevinsky and Tadić classification of the admissible dual ([24], [18]), the proofs here rely on [1] (dual of an irreducible representation is irreducible), [12] and [7] (algorithm for computing the dual) and some easy tricks from [16], [4] and [10], and do not involve any complicated technique.

The idea of searching for "simple proof" for classification of unitary representations, together with a list of basic tricks to use, are due to Marko Tadić ([19] for example). He also was the first to formulate some properties of Speh representations (*formula for ends of complimentary series, character formula, Speh representations are prime elements in the ring of representations, dual of a Speh representation is Speh*) and to prove them when $D = F$. The starting point of my proof here of the assertion *Speh representations are unitary* is also due to Tadić who found the simple but brilliant trick reducing the problem of unitarity to a problem of irreducibility.

I would like to thank Guy Henniart who read the paper and made useful observations.

*Notation.* — Let $(F, |\ |_F)$ be a local non archimedean field of any characteristic $ch(F)$ and $D$ a central division algebra of finite dimension $d^2$ over $F$. Let $\nu_n : GL_n(D) \to \mathbb{C}^\times$ be the character $g \mapsto |\det(g)|_F$, denoted simply $\nu$ when no confusion may occur (det is the reduced norm). All representations here will be admissible of finite length. If $\pi$, $\pi'$ are representations of $GL_a(D)$ and $GL_b(D)$

such that $a+b=n$, let $P$ be the parabolic subgroup of $GL_n(D)$ containing the upper triangular matrices and of Levi factor $GL_a(D)\times GL_b(D)$ and set $\pi\times\pi' := \mathrm{ind}_P^{GL_n(D)}\pi\otimes\pi'$.

The results in this section for which no other reference is given may be found in [24] (for $D=F$) and [18] (for general $D$). In [18] the characteristic zero is assumed, but this restriction may be removed ([5, 2.2]). Let $Gr_n$ denote the Grothedieck group of admissible representations of finite length of $GL_n(D)$, and, by convention, $Gr_0=\mathbb{Z}$. Set $\mathscr{R} := \oplus_{n\in\mathbb{N}}Gr_n$. The composition law $\times$ induces a composition law $*$ in $\mathscr{R}$ which makes $\mathscr{R}(+,*)$ into a commutative $\mathbb{Z}$-algebra ($\times$ is not commutative, but $*$ is). We write *representation* for *isomorphy class of representations* where no confusion may occur. The set Irr of irreducible representations of $GL_n(D)$, $n\in\mathbb{N}^\times$, is a natural linear basis of $\mathscr{R}$.

Let $\delta$ be an essentially square integrable (i.e. twist of a square integrable by a character) representation of $GL_n(D)$. Let $s(\delta)$ be the smallest positive real number $x$ such that $\delta\times\nu^x\delta$ is reducible. Then $s(\delta)\in\mathbb{N}^*$ and $s(\delta)|d$ (so $s(\delta)=1$ if $D=F$). Set $\nu_\delta := \nu^{s(\delta)}$. Then $\delta$ may be obtained as the unique irreducible subrepresentation of a representation of type $\nu_\delta^a\rho\times\nu_\delta^{a-1}\rho\times\nu_\delta^{a-2}\rho\times\cdots\times\nu_\delta^{a-m+1}\rho$, with $a\in\mathbb{R}$, $m$ a divisor of $n$, and $\rho$ a cuspidal unitary representation of $GL_{\frac{n}{m}}(D)$. Also, $a,m$ and $\rho$ are determined by $\delta$. We then write $\delta=Z(\rho,a,m)$. We have $s(\rho)=s(\delta)$. We set $e(\delta) := a-\frac{m-1}{2}$. Then $\delta$ is unitary (i.e. is square integrable) if and only if $e(\delta)=0$. In this case we simply write $\delta=Z(\rho,m)$, understood as: $\delta$ is the unique subrepresentation of $\nu_\delta^{\frac{m-1}{2}}\rho\times\nu_\delta^{\frac{m-1}{2}-1}\rho\times\nu_\delta^{\frac{m-1}{2}-2}\rho\times\cdots\times\nu_\delta^{-\frac{m-1}{2}}\rho$. We make the convention that $Z(\rho,a,0)$ is the unit element of $\mathscr{R}$.

To any irreducible representation $\pi$ of $GL_n(D)$ we may associate a unique multiset $[\delta_1,\delta_2,\ldots,\delta_k]$ such that

- $\delta_i$ is an essentially square integrable representations of some $GL_{n_i}(D)$, $n_i\geq 1$,
- $\sum_i n_i = n$ and
- if we reorder the $d_i$ such that $e(\delta_i)$ is decreasing with $i$, then $\pi$ is the unique irreducible quotient of $\delta_1\times\delta_2\times\cdots\times\delta_k$.

We write $\pi=L([\delta_1,\delta_2,\ldots,\delta_k])$. We call *standard representation* an element of $\mathscr{R}$ which is a product $\delta_1*\delta_2*\cdots*\delta_k$ with $d_i$ essentially square integrable representations. It is known that $\delta_1*\delta_2*\cdots*\delta_k$ determines the multiset $[\delta_1,\delta_2,\ldots,\delta_k]$, so we may write $\pi=L(\delta_1*\delta_2*\cdots*\delta_k)$ instead of $\pi=L([\delta_1,\delta_2,\ldots,\delta_k])$. Let $\mathscr{D}$ be the set of essentially square integrable representations in $\mathscr{R}$ and Std be the set of standard representations. Then $L$ realizes a bijection from Std to Irr (and its restriction to $\mathscr{D}$ is the identity). If $\Theta$ is a standard representation and $\pi=L(\Theta)$, we then say that $\pi$ is *the Langlands quotient* of $\Theta$.