

Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

FLIPS OF MODULI OF STABLE TORSION FREE SHEAVES WITH $c_1 = 1$ ON \mathbb{P}^2

Ryo Ohkawa

Tome 142
Fascicule 3

2014

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre national de la recherche scientifique
pages 349-378

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 3, tome 142, septembre 2014

Comité de rédaction

Jean BARGE	Daniel HUYBRECHTS
Gérard BESSON	Yves LE JAN
Emmanuel BREUILLARD	Julien MARCHÉ
Antoine CHAMBERT-LOIR	Laure SAINT-RAYMOND
Jean-François DAT	Wilhelm SCHLAG
Charles FAVRE	
Raphaël KRIKORIAN (dir.)	

Diffusion

Maison de la SMF Case 916 - Luminy 13288 Marseille Cedex 9 France smf@smf.univ-mrs.fr	Hindustan Book Agency O-131, The Shopping Mall Arjun Marg, DLF Phase 1 Gurgaon 122002, Haryana Inde	AMS P.O. Box 6248 Providence RI 02940 USA www.ams.org
--------------------------------------------------------------------------------------------------------------------------------------------	-----------------------------------------------------------------------------------------------------------------	----------------------------------------------------------------------------------------------------

Tarifs

Vente au numéro : 43 € (\$ 64)
Abonnement Europe : 300 €, hors Europe : 334 € (\$ 519)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • <http://smf.emath.fr/>

© Société Mathématique de France 2014

Tous droits réservés (article L 122-4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335-2 et suivants du CPI.

ISSN 0037-9484

Directeur de la publication : Marc PEIGNÉ

FLIPS OF MODULI OF STABLE TORSION FREE SHEAVES WITH $c_1 = 1$ ON \mathbb{P}^2

BY RYO OHKAWA

Dedicated to Takao Fujita on the occasion of his 60th birthday

ABSTRACT. — We study flips of moduli schemes of stable torsion free sheaves E with $c_1(E) = 1$ on \mathbb{P}^2 via wall-crossing phenomena of Bridgeland stability conditions. They are described as stratified Grassmann bundles by a variation of stability of modules over certain finite dimensional algebra.

RÉSUMÉ (Flips de modules de faisceaux stables et sans torsion avec $c_1 = 1$ sur \mathbb{P}^2)

Nous étudions des flips de schémas de modules de faisceaux stables et sans torsion E avec $c_1(E) = 1$ sur \mathbb{P}^2 à travers des phénomènes de traversée de mur des conditions de stabilité de Bridgeland. Ils sont décrits en tant que fibrés grassmanniens par une variation de stabilité de modules au-dessus d'une certaine algèbre de dimension finie.

Texte reçu le 13 septembre 2010, révisé le 10 janvier 2012, accepté le 7 septembre 2012.

RYO OHKAWA, Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 JAPAN • E-mail : ohkawa@kurims.kyoto-u.ac.jp

2010 Mathematics Subject Classification. — 18E30, 14D20.

Key words and phrases. — Bridgeland stability, moduli of vector bundles.

The author is grateful to his adviser Takao Fujita for many valuable comments and the encouragement. He would like to thank Hiraku Nakajima and Kōta Yoshioka for useful discussions and valuable comments, which gave him a motivation to write this paper. He would also like to thank Minoru Itoh and Andrew Obus for corrections of English and suggestions to make the paper more readable, and the referees for careful readings and pointing out a gap in Appendix A in the preliminary version of the paper. The author completed this paper during his stay in the Max Planck Institute for Mathematics. He appreciates the hospitality of the institute. This research was supported in part by JSPS Global COE program “Computationism As a Foundation of the Sciences”.

1. Introduction

1.1. Background. — We denote by $M_{\mathbb{P}^2}(r, c_1, n)$ the moduli scheme of semistable torsion free sheaves E on \mathbb{P}^2 with the Chern class $c(E) = (r, c_1, n) \in H^{2*}(\mathbb{P}^2, \mathbb{Z})$. In this paper we treat the case where $c_1 = 1$. Then semistability and stability for E coincide. When $n \geq r \geq 2$, or $r = 1$ and $n \geq 2$, the Picard number of $M_{\mathbb{P}^2}(r, 1, n)$ is equal to 2 and we have two birational morphisms from $M_{\mathbb{P}^2}(r, 1, n)$, which are described below.

One is defined by J. Li [12] for general cases. We denote by $M_{\mathbb{P}^2}(r, 1, n)_0$ the open subset of $M_{\mathbb{P}^2}(r, 1, n)$ consisting of stable *vector bundles*. The Uhlenbeck compactification $\overline{M}_{\mathbb{P}^2}(r, 1, n)$ of $M_{\mathbb{P}^2}(r, 1, n)_0$ is described set-theoretically by

$$\overline{M}_{\mathbb{P}^2}(r, 1, n) = \bigsqcup_{i \geq 0} (M_{\mathbb{P}^2}(r, 1, n - i)_0 \times S^i(\mathbb{P}^2)).$$

The map $\pi: M_{\mathbb{P}^2}(r, 1, n) \rightarrow \overline{M}_{\mathbb{P}^2}(r, 1, n)$, $E \mapsto \pi(E)$ is defined by

$$\pi(E) = (E^{**}, \text{Supp}(E^{**}/E)) \in M_{\mathbb{P}^2}(r, 1, n - i)_0 \times S^i(\mathbb{P}^2),$$

where E^{**} is the double dual of E and i is the length of E^{**}/E .

In the case where $r = 1$, this morphism is called the Hilbert-Chow morphism $\pi: (\mathbb{P}^2)^{[n]} \rightarrow S^n(\mathbb{P}^2)$. In the case where $r \geq 2$, this map is also birational since it is an isomorphism on $M_{\mathbb{P}^2}(r, 1, n)_0$ to its image. It is shown that the codimension of the complement of $M_{\mathbb{P}^2}(r, 1, n)_0$ is equal to 1 when we have $M_{\mathbb{P}^2}(r, 1, n - 1) \neq \emptyset$ (cf. [13, Proposition 3.23]). Hence this map is a divisorial contraction.

The other birational morphism is defined by Yoshioka. In his paper [17] on moduli of torsion free sheaves on rational surfaces, he studied the following morphism:

$$\psi: M_{\mathbb{P}^2}(r, 1, n) \rightarrow M_{\mathbb{P}^2}(n + 1, 1, n).$$

For any $E \in M_{\mathbb{P}^2}(r, 1, n)$, $\psi(E)$ is defined by the exact sequence

$$(1) \quad 0 \rightarrow \text{Ext}_{\mathbb{P}^2}^1(E, \mathcal{O}_{\mathbb{P}^2})^* \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow \psi(E) \rightarrow E \rightarrow 0,$$

which is called the universal extension, where $\text{Ext}_{\mathbb{P}^2}^1(E, \mathcal{O}_{\mathbb{P}^2})^*$ is the dual vector space of $\text{Ext}_{\mathbb{P}^2}^1(E, \mathcal{O}_{\mathbb{P}^2})$. Here we have $\text{Hom}_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2}) = \text{Ext}_{\mathbb{P}^2}^2(E, \mathcal{O}_{\mathbb{P}^2}) = 0$ and $(n + 1, 1, n) \in H^{2*}(\mathbb{P}^2, \mathbb{Z})$ is the Chern class of $[E] - \chi(E, \mathcal{O}_{\mathbb{P}^2})[\mathcal{O}_{\mathbb{P}^2}] = [E] + \dim \text{Ext}_{\mathbb{P}^2}^1(E, \mathcal{O}_{\mathbb{P}^2})[\mathcal{O}_{\mathbb{P}^2}] \in K(\mathbb{P}^2)$, where

$$\chi(E, \mathcal{O}_{\mathbb{P}^2}) = \sum_i (-1)^i \dim_{\mathbb{C}} \text{Ext}_{\mathbb{P}^2}^i(E, \mathcal{O}_{\mathbb{P}^2}).$$

Furthermore, the moduli space $M_{\mathbb{P}^2}(r, 1, n)$ has a stratification

$$M_{\mathbb{P}^2}(r, 1, n) = \bigsqcup_{i=0}^r M_{\mathbb{P}^2}^i(r, 1, n),$$

where $M_{\mathbb{P}^2}^i(r, 1, n) := \{E \in M_{\mathbb{P}^2}(r, 1, n) \mid \dim_{\mathbb{C}} \text{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}, E) = i\}$ is called the *Brill-Noether locus*. The following theorem is shown in [17].

THEOREM 1.1 ([17, Theorem 5.8]). — (i) *There exists an isomorphism*

$$M_{\mathbb{P}^2}^i(r, 1, n) \cong \psi^{-1}(M_{\mathbb{P}^2}^{n-r+i+1}(n+1, 1, n)).$$

(ii) *The restriction of ψ to each stratum $M_{\mathbb{P}^2}^i(r, 1, n)$ is a $\text{Gr}(n-r+i+1, i)$ -bundle over the stratum $M_{\mathbb{P}^2}^{n-r+i+1}(n+1, 1, n)$.*

By the above theorem, if $n > r+2$, ψ is a birational morphism to the image $\text{im } \psi$, and it is a flipping contraction. By the theory of birational geometry [3], we have the diagram called flip:

$$(2) \quad M_+(r, 1, n) \leftarrow \cdots \rightarrow M_{\mathbb{P}^2}(r, 1, n)$$

$$\begin{array}{ccc} & & \\ & \searrow \psi_+ & \swarrow \psi \\ & \text{im } \psi & \end{array}$$

The purpose of this note is to describe spaces $M_+(r, 1, n)$, $\text{im } \psi$ and the morphism ψ_+ in the above diagram in terms of moduli spaces. We follow ideas in [15]. We consider $M_{\mathbb{P}^2}(r, 1, n)$ as a moduli scheme of semistable modules over the finite dimensional algebra

$$B := \text{End}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \Omega_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$$

via Bridgeland stability conditions on $D^b(\mathbb{P}^2)$. This enables us to study the wall-crossing phenomena of the moduli scheme as the stability changes by using the result of [15].

1.2. Main results. — We introduce the exceptional collection

$$\mathfrak{E} := (\mathcal{O}_{\mathbb{P}^2}(1), \Omega_{\mathbb{P}^2}^1(3), \mathcal{O}_{\mathbb{P}^2}(2))$$

on \mathbb{P}^2 and put $\mathcal{E} := \mathcal{O}_{\mathbb{P}^2}(1) \oplus \Omega_{\mathbb{P}^2}^1(3) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ and $B := \text{End}_{\mathbb{P}^2}(\mathcal{E})$. We denote abelian categories of coherent sheaves on \mathbb{P}^2 and finitely generated right B -modules by $\text{Coh}(\mathbb{P}^2)$ and $\text{mod-}B$, respectively. Then by Bondal's Theorem [4], the functor $\Phi := \mathbf{R} \text{Hom}_{\mathbb{P}^2}(\mathcal{E}, -)$ gives an equivalence

$$\Phi: D^b(\mathbb{P}^2) \cong D^b(B),$$

where $D^b(\mathbb{P}^2)$ and $D^b(B)$ are bounded derived categories of $\text{Coh}(\mathbb{P}^2)$ and $\text{mod-}B$, respectively. The equivalence Φ also induces an isomorphism $\varphi: K(\mathbb{P}^2) \cong K(B)$ between Grothendieck groups of $\text{Coh}(\mathbb{P}^2)$ and $\text{mod-}B$.

For a class α in $K(B)$, we put

$$\alpha^\perp := \{\theta \in \text{Hom}_{\mathbb{Z}}(K(B), \mathbb{R}) \mid \theta(\alpha) = 0\}.$$