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HYPERELLIPTIC d-OSCULATING COVERS AND RATIONAL SURFACES

by Armando Treibich

Abstract. — Let d be a positive integer, $\mathbb{K}$ an algebraically closed field of characteristic $\mathbf{p} \neq 2$ and X an elliptic curve defined over $\mathbb{K}$. We consider the hyperelliptic curves equipped with a projection over X, such that the natural image of X in the Jacobian of the curve osculates to order d to the embedding of the curve, at a Weierstrass point. We first study the relations between the degree n, the arithmetic genus g and the osculating degree d of such covers. We prove that they are in a one-to-one correspondence with rational curves of linear systems in a rational surface and deduce $(d-1)$-dimensional families of hyperelliptic d-osculating covers, of arbitrary big genus g if $\mathbf{p}=0$ or such that $2g < \mathbf{p}(2d+1)$ if $\mathbf{p} > 2$. It follows at last, $(g+d-1)$-dimensional families of solutions of the KdV hierarchy, doubly periodic with respect to the d-th variable.

Résumé (*Projections hyperelliptiques d-osculantes et surfaces rationnelles*)
Soit d un entier positif, $\mathbb{K}$ un corps algébriquement clos de caractéristique $\mathbf{p} \neq 2$ et X une courbe elliptique définie sur $\mathbb{K}$. On étudie les courbes hyperelliptiques munies d'une projection sur X, telles que l'image naturelle de X dans la jacobienne de la courbe, oscule à l'ordre d au plongement de celle-ci, et ce en un point de Weierstrass. On étudie tout d'abord les relations entre le degré n, le genre arithmétique g et l'ordre d'osculation d des ces projections. On prouve qu'elles sont en correspondance biunivoque avec des courbes rationnelles dans des systèmes linéaires d'une surface rationnelle et on en déduit des familles $(d-1)$-dimensionnelles de revetements hyperelliptiques d-osculants de genre g, arbitrairement grand si la caractéristique $\mathbf{p}=0$, ou $2g < \mathbf{p}(2d+1)$ si $p\mathbf{p} > 2$.

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Il en résulte des familles $(g+d-1)$-dimensionnelles de solutions de la hiérarchie KdV, doublement périodiques par rapport à la d-ième variable.

1. Introduction

1.1. — Let $\mathbb{P}^1 := \mathbb{K} \cup \{\infty\}$ and (X, q) denote, respectively, the projective line and a fixed elliptic curve marked at its origin, both defined over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic $p \neq 2$. We will study all finite separable marked morphisms $\pi : (\Gamma, p) \to (X, q)$, called hereafter *hyperelliptic covers*, such that Γ is a degree-2 cover of $\mathbb{P}^1$, ramified at the smooth point $p \in \Gamma$. Canonically associated to π there is the Abel (rational) embedding of Γ into its *generalized Jacobian*, $A_p : \Gamma \to \mathrm{Jac}\Gamma$, and $\{0\} \subsetneq V^1_{\Gamma,p} \cdots \subsetneq V^g_{\Gamma,p}$, the flag of hyperosculating planes to $A_p(\Gamma)$ at $A_p(p) \in \mathrm{Jac}\Gamma$ (cf. 2.1 & Definition 2.1). On the other hand, we also have the homomorphism $\iota_\pi : X \to \mathrm{Jac}\Gamma$, obtained by dualizing π. There is a smallest positive integer d such that the tangent line to $\iota_\pi(X)$ is contained in the d-dimensional osculating plane $V^d_{\Gamma,p}$. We call it *the osculating order* of π, and π a *hyperelliptic d-osculating cover* (Definition 2.3 (2)). If π factors through another *hyperelliptic cover*, the arithmetic genus increases, while the *osculating order* can not decrease (Corollary 2.7).

Studying, characterizing and constructing those with given *osculating order d* but maximal possible arithmetic genus, so-called *minimal-hyperelliptic d-osculating covers*, will be one of the main issues of this article. The other one, to which the first issue reduces, is the construction of all rational curves in a particular anticanonical rational surface associated to X (i.e., a rational surface with an effective anticanonical divisor). Both problems are interesting on their own and in any characteristic $p \neq 2$. Up to recently they were only considered over the complex numbers and through their link with solutions of the *Korteweg-deVries* hierarchy, doubly periodic with respect to the d-th KdV flow (cf. [1], [3], [8], [10], [16], [17] for $d = 1$, [13], [2], [4], [5] for $d = 2$ and [15] for $d \geq 3$). At last let me point out a less restrictive situation already studied but only over $\mathbb{C}$: one may drop the hyperelliptic assumption on the curve Γ and choose a degree-d effective divisor $D = \sum_{j=1}^{l} m_j p_j$ with support at l generic points of Γ. Forcing the line $\iota_\pi(X)$ to be contained in the hyperosculating plane $\sum_{j=1}^{l} V^{m_j}_{\Gamma,p_j}$ one obtains so-called D-tangential covers. In case $d = 1$, or $m_j = 1$ and π is étale at p_j ($\forall j = 1, \cdots, l$), the corresponding covers were constructed by solving the associated Calogero-Moser integrable system (e.g., [11] & [10]; see also [2]) and give rise to $d \times d$ matrix solutions of the Kadomtsev-Petviashvili equation, a suitable generalization of the KdV one (cf. [9]). Constructing the general D-tangential cover was considered as a geometrical problem for Generalized Jacobians of irreducible complex projective curves but (reduced to and) solved as a geometric problem for a particular ruled surface $\pi_S : S \to X$ (cf. [14]), the same one constructed in Section 3 below.

We sketch hereafter the structure and main results of our article. (1)

1. We start defining in Section 2 the Abel rational embedding $A_p : \Gamma \to \mathrm{Jac}\Gamma$, and construct the flag $\{0\} \subsetneq V^1_{\Gamma,p} \cdots \subsetneq V^g_{\Gamma,p} = H^1(\Gamma, O_\Gamma)$, of *hyperosculating planes* at the image of any smooth point $p \in \Gamma$. We then define the homomorphism $\iota_\pi : X \to \mathrm{Jac}\Gamma$, canonically associated to the *hyperelliptic cover* π, and its *osculating order* (Definition 2.3 (2)). Regardless of the *osculating order*, we prove that any degree-n *hyperelliptic cover* has odd ramification index at the marked point, say ρ, and factors through a unique one of maximal arithmetic genus $2n - \frac{\rho+1}{2}$ (Theorem 2.5). We finish characterizing the *osculating order* by the existence of a particular projection $\kappa : \Gamma \to \mathbb{P}^1$ (Theorem 2.5).

2. The d-osculating criterion in Theorem 2.5 paves the way to the algebraic surface approach developed in the remaining sections. The main characters are played by (two morphisms between) three projective surfaces, canonically associated to the elliptic curve X:
   - $e : S^\perp \to S$: the blowing-up of a particular ruled surface $\pi_S : S \to X$, at the 8 fixed points of its involution;
   - $\varphi : S^\perp \to \widetilde{S}$: a projection onto an anticanonical rational surface.
3. Once S, $S^\perp$ and $\widetilde{S}$ are constructed (Definitions 3.1 & 3.3), we prove that any *hyperelliptic d-osculating cover* $\pi : (\Gamma, p) \to (X, q)$ factors canonically through a curve $\Gamma^\perp \subset S^\perp$, and projects, via $\varphi : S^\perp \to \widetilde{S}$, onto a rational irreducible curve $\widetilde{\Gamma} \subset \widetilde{S}$ (Proposition 3.7). We also prove that any *hyperelliptic d-osculating cover* dominates a unique one of same *osculating order* d, but maximal arithmetic genus, so-called *minimal-hyperelliptic* (Corollary 3.8). Conversely, given $\widetilde{\Gamma} \subset \widetilde{S}$, we study when and how one can recover all *minimal-hyperelliptic d-osculating covers* having same canonical projection $\widetilde{\Gamma}$ (Corollary 3.10).
4. Section 4 is mainly devoted to studying the linear equivalence class of the curve $\Gamma^\perp \subset S^\perp$, canonically associated to any *hyperelliptic d-osculating cover* π, and associated invariants (Lemma 4.2 & Theorem 4.3). We end up with a numerical characterization of *minimal-hyperelliptic d-osculating covers* (Corollary 4.5).

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