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HÖLDER CONTINUITY OF LYAPUNOV EXPONENT FOR QUASI-PERIODIC JACOBI OPERATORS

BY KAI TAO

ABSTRACT. — We consider the quasi-periodic Jacobi operator $H_{x,\omega}$ in $l^2(\mathbb{Z})$: $(H_{x,\omega}\phi)(n) = -b(x+(n+1)\omega)\phi(n+1) - b(x+n\omega)\phi(n-1) + a(x+n\omega)\phi(n) = E\phi(n)$, $n \in \mathbb{Z}$, where $a(x)$, $b(x)$ are analytic functions on $\mathbb{T}$, b is not identically zero, and ω obeys some strong Diophantine condition. We consider the corresponding unimodular cocycle. We prove that if the Lyapunov exponent $L(E)$ of the cocycle is positive for some $E = E_0$, then there exist $\rho_0 = \rho_0(a, b, \omega, E_0)$, $\beta = \beta(a, b, \omega)$ such that $|L(E) - L(E')| < |E - E'|^\beta$ for any $E, E' \in (E_0 - \rho_0, E_0 + \rho_0)$. If $L(E) > 0$ for all E in some compact interval I, then $L(E)$ is Hölder continuous on I with Hölder exponent $\beta = \beta(a, b, \omega, I)$. In our derivation we follow the refined version of the Goldstein-Schlag method [3] developed by Bourgain and Jitomirskaya [2].

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1. Introduction

We consider the quasi-periodic Jacobi operator $H_{x,\omega}$ in $l^2(\mathbb{Z})$:

$$\begin{aligned}(1.1)\quad &(H_{x,\omega}\phi)(n)\\ &= -b(x+(n+1)\omega)\phi(n+1) - b(x+n\omega)\phi(n-1) + a(x+n\omega)\phi(n)\\ &= E\phi(n),\ n\in\mathbb{Z},\end{aligned}$$

where $a(x)$, $b(x)$ are real analytic functions on $\mathbb{T}$, b is not identically zero. Set

$$A(x,E,\omega) = \frac{1}{b(x+\omega)}\begin{pmatrix} a(x)-E & -b(x)\\ b(x+\omega) & 0\end{pmatrix}.$$

$$\begin{aligned}M_N(x,E,\omega) &= M_{[1,N]}(x,E,\omega)\\ &= A(x+(N-1)\omega,E,\omega)A(x+(N-2)\omega,E,\omega)\cdots A(x,E,\omega).\end{aligned}$$

Define the unimodular matrix

$$\tilde{M}_N(x,E,\omega) = \tilde{M}_{[1,N]}(x,E,\omega) := \frac{M_{[1,N]}(x,E,\omega)}{|\det M_{[1,N]}(x,E,\omega)|^{\frac{1}{2}}}.$$

As

$$\det A(x,E,\omega) = \frac{b(x)}{b(x+\omega)},$$

then

$$(1.2)\qquad \det M_{[1,N]}(x,E,\omega) = \prod_{n=0}^{N-1}\frac{b(x+n\omega)}{b(x+(n+1)\omega)} = \frac{b(x)}{b(x+N\omega)},$$

and

$$(1.3)\quad \log\|\tilde{M}_{[1,N]}(x,E,\omega)\| = \log\|M_{[1,N]}(x,E,\omega)\| - \frac{1}{2}\log\Big|\frac{b(x)}{b(x+N\omega)}\Big|.$$

Remark 1.1. — (1) Note that

$$\|A(x,E,\omega)\| \le \frac{C(a,b,E)}{|b(x+\omega)|},$$

where the constant $C(a,b,E)$ satisfies

$$C(a,b,E_0) = \sup_{|E|\le E_0} C(a,b,E) < +\infty.$$

Therefore,

$$\frac{1}{N}\log\|M_{[1,N]}(x,E,\omega)\| \le \log C(a,b,E) - \frac{1}{N}\sum_{n=1}^{N}\log|b(x+n\omega)|.$$

In this paper we always assume that $|E| \leq E_0$, where E_0 depends on a, b. For that matter we suppress E from the notation of some of the constants involved.

(2) $\log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \geq 0$, since $\tilde{M}_{[1,N]}(x, E, \omega)$ is unimodular.

(3)
$$\begin{aligned} 0 &\leq \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \\ &= \frac{1}{N} \log \|M_{[1,N]}(x, E, \omega)\| - \frac{1}{2N} \log |\frac{b(x)}{b(x+N\omega)}| \\ &\leq \log C(a, b, E) - \frac{1}{N} \sum_{n=1}^{N} \log |b(x+n\omega)| - \frac{1}{2N} \log |\frac{b(x)}{b(x+N\omega)}|. \end{aligned}$$

(4) It is a well-known fact that if b is an analytic function not identically zero, then $(\log |b|)^2$ is integrable. Set
$$D = \int_{\mathbb{T}} \log |b(\theta)| d\theta.$$

Therefore,
$$\begin{aligned} \int_{\mathbb{T}} \left| \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \right| dx &= \int_{\mathbb{T}} \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| dx \\ &\leq C'(a, b) - D := C''(a, b). \end{aligned}$$

Similarly,
$$\int_{\mathbb{T}} (\frac{1}{N} \log \|\tilde{M}_N(x, E, \omega)\|)^2 dx \leq \tilde{C}(a, b).$$

(5) Combining (4) with (1.3), we conclude that $\frac{1}{N} \log \|M_{[1,N]}(x, E, \omega)\|$ is integrable, and
$$\frac{1}{N} \int_{\mathbb{T}} \log \|M_N(x, E, \omega)\| dx = \frac{1}{N} \int_{\mathbb{T}} \log \|\tilde{M}_N(x, E, \omega)\| dx.$$

Set
$$L_N(E, \omega) = \frac{1}{N} \int_{\mathbb{T}} \log \|M_N(x, E, \omega)\| dx = \frac{1}{N} \int_{\mathbb{T}} \log \|\tilde{M}_N(x, E, \omega)\| dx.$$

Note that $L_N(E, \omega) > 0$. Set
$$B(x, E, \omega) := \begin{pmatrix} a(x) - E & -b(x) \\ b(x+\omega) & 0 \end{pmatrix},$$

and
$$\begin{aligned} T_N(x, E, \omega) &= T_{[1,N]}(x, E, \omega) \\ &:= B(x + (N-1)\omega, E, \omega) B(x + (N-2)\omega, E, \omega) \cdots B(x, E, \omega). \end{aligned}$$

Then

$$M_{[1,N]}(x,E,\omega) = T_{[1,N]}(x,E,\omega) \prod_{n=N-1}^{0} \frac{1}{b(x+(n+1)\omega)},$$

$$\begin{aligned} \tilde{M}_{[1,N]}(x,E,\omega) &= \frac{|b(x+N\omega)|^{\frac{1}{2}}}{|b(x)|^{\frac{1}{2}}} M_{[1,N]}(x,E,\omega) \\ &= \frac{|b(x+N\omega)|^{\frac{1}{2}}}{|b(x)|^{\frac{1}{2}}} \prod_{n=0}^{N-1} \frac{1}{b(x+(n+1)\omega)} T_{[1,N]}(x,E,\omega), \end{aligned} \tag{1.4}$$

$$\|\tilde{M}_{[1,N]}(x,E,\omega)\| = \prod_{n=0}^{N-1} \frac{1}{|b(x+n\omega)b(x+(n+1)\omega)|^{\frac{1}{2}}} \|T_{[1,N]}(x,E,\omega)\|, \tag{1.5}$$

and

$$\log \|\tilde{M}_{[1,N]}(x,E,\omega)\| = \log \|T_{[1,N]}(x,E,\omega)\| - \frac{1}{2} \sum_{n=0}^{N-1} \log |b(x+n\omega)b(x+(n+1)\omega)|.$$

Note also for future reference that

$$|\det T_{[1,N]}(x,E,\omega)| = \prod_{n=0}^{N-1} |b(x+n\omega)||b(x+(n+1)\omega)|. \tag{1.6}$$

Combining (1) with Remark 1.1, we conclude that $\frac{1}{N} \log \|T_{[1,N]}(x,E,\omega)\|$ is integrable,

$$J_N(E,\omega) := \frac{1}{N} \int_{\mathbb{T}} \log \|T_{[1,N]}(x,\omega)\| dx = L_N(E,\omega) + D. \tag{1.7}$$

Due to the subadditive property, the limits

$$\begin{aligned} L(E,\omega) &= \lim_{N\to\infty} \int_{\mathbb{T}} \frac{1}{N} \log \|M_N(x,E,\omega)\| dx = \lim_{N\to\infty} \int_{\mathbb{T}} \frac{1}{N} \log \|\tilde{M}_N(x,E,\omega)\| dx \\ &= \lim_{N\to\infty} L_N(E,\omega), \end{aligned} \tag{1.8}$$

$$J(E,\omega) = \lim_{N\to\infty} \int_{\mathbb{T}} \frac{1}{N} \log \|T_N(x,E,\omega)\| dx = \lim_{N\to\infty} J_N(E,\omega) = L(E,\omega) + D \tag{1.9}$$

exist. Moreover, $L(E,\omega) \geq 0$. Fix some $\alpha > 1$. Throughout this paper we assume that $\omega \in (0,1)$ satisfies the Diophantine condition

$$\|n\omega\| \geq \frac{C_\omega}{n(\log n)^\alpha} \quad \text{for all } n. \tag{1.10}$$

It is well known that for a fixed $\alpha > 1$ almost every ω satisfies (1.10).

The main theorem in this paper is