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ESTIMATES OF THE LINEARIZATION OF CIRCLE DIFFEOMORPHISMS

BY MOSTAPHA BENHENDA

ABSTRACT. — A celebrated theorem by Herman and Yoccoz asserts that if the rotation number α of a C^∞-diffeomorphism of the circle f satisfies a Diophantine condition, then f is C^∞-conjugated to a rotation. In this paper, we establish explicit relationships between the C^k norms of this conjugacy and the Diophantine condition on α. To obtain these estimates, we follow a suitably modified version of Yoccoz's proof.

RÉSUMÉ (*Estimées de la linéarisation de difféomorphismes du cercle*)
Un célèbre théorème de Herman et Yoccoz affirme que si le nombre de rotation α d'un C^∞-difféomorphisme du cercle f satisfait une condition diophantienne, alors f est C^∞-conjugué à une rotation. Dans cet article, nous établissons des relations explicites entre les C^k normes de cette conjuguée et la condition diophantienne sur α. Pour obtenir ces estimées, nous suivons une version convenablement modifiée de la preuve de Yoccoz.

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1. Introduction

In his seminal work, M. Herman [5] shows the existence of a set A of Diophantine numbers of full Lebesgue measure such that for any circle diffeomorphism f of class C^ω (resp. C^∞) of rotation number $\alpha \in A$, there is a C^ω-diffeomorphism (resp. C^∞-diffeomorphism) h such that $hfh^{-1} = R_\alpha$. In the C^∞ case, J. C. Yoccoz [14] extended this result to all Diophantine rotation numbers. Results in analytic class and in finite differentiability class subsequently enriched the global theory of circle diffeomorphisms [9, 8, 7, 13, 6, 15, 4, 10]. In the perturbative theory, KAM theorems usually provide a bound on the norm of the conjugacy that involves the norm of the perturbation and the Diophantine constants of the number α (see [5, 12, 11] for example). We place ourselves in the global setting, we compute a bound on the norms of this conjugacy h in function of the class of differentiability k, of norms of f, and of the Diophantine parameters β and C_d of α (an irrational number $\alpha \in DC(C_d, \beta)$ satisfies a Diophantine condition of order $\beta \geq 0$ and constant $C_d > 0$ if for any rational number p/q, we have: $|\alpha - p/q| \geq C_d/q^{2+\beta}$). The dependency in C_d is particularly interesting to study, because for any fixed $\beta > 0$, the set of Diophantine numbers of parameter β has full Lebesgue measure. It follows that the control of the conjugacy for a typical diffeomorphism, with fixed norms, is approached as $C_d \to 0$.

To obtain these estimates, we follow a suitably modified version of Yoccoz's proof. Indeed, Yoccoz's proof needs to be modified because a priori, it does not exclude the fact that the following set could be unbounded for any fixed $X > 0$:

$$\begin{aligned} E_X = \{|Dh|_0 \,/ \exists f \in \mathrm{Diff}^k_+(\mathbb{T}^1),\ f = h^{-1} R_\alpha h,& \\ \alpha \in DC(\beta, C_d), \max(k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}) \leq X\}& \end{aligned}$$

where $\mathrm{Diff}^k_+(\mathbb{T}^1)$ denotes the group of orientation-preserving circle diffeomorphisms of class C^k, Df denotes the derivative of f, $W(f)$ the total variation of $\log Df$, and Sf the Schwarzian derivative of f.

These estimates have natural applications to the global study of circle diffeomorphisms with Liouville rotation number: in [2], they allow to show the following results: 1) there is a Baire-generic set $A_1 \subset \mathbb{R}$ such that for any $f \in D^\infty(\mathbb{T}^1)$ of rotation number $\alpha \in A_1$, there is a sequence $h_n \in D^\infty(\mathbb{T}^1)$ such that $h_n^{-1} f h_n \to R_\alpha$ in the C^∞-topology. 2) There is a Baire-generic set $A_2 \subset \mathbb{R}$ such that for any $f \in D^\infty(\mathbb{T}^1)$ of rotation number $\alpha \in A_2$ and any g of class C^∞ with $fg = gf$, f and g are accumulated in the C^∞-topology by commuting C^∞-diffeomorphisms that are C^∞-conjugated to rotations. Moreover, if β is the rotation number of g, R_α and R_β are accumulated in the C^∞-topology by commuting C^∞-diffeomorphisms that are C^∞-conjugated to f and g.

1.1. Notations. — We follow the notations of [14].

- The circle is denoted by $\mathbb{T}^1$. The group of $\mathbb{Z}$-periodic maps of class C^r of the real line is denoted by $C^r(\mathbb{T}^1)$. We work in $D^r(\mathbb{T}^1)$, which is the group of diffeomorphisms f of class C^r of the real line such that $f - Id \in C^r(\mathbb{T}^1)$. It is the universal cover of the group of orientation-preserving circle diffeomorphisms of class C^r. Note that if $f \in D^r(\mathbb{T}^1)$ and $r \geq 1$, then $Df \in C^{r-1}(\mathbb{T}^1)$.
- The Schwarzian derivative Sf of $f \in D^3(\mathbb{T}^1)$ is defined by:

$$Sf = D^2 \log Df - \frac{1}{2}(D \log Df)^2.$$

- The total variation of the logarithm of the first derivative of f is:

$$W(f) = \sup_{0 \leq a_0 \leq \cdots \leq a_n \leq 1} \sum_{i=0}^{n} |\log Df(a_{i+1}) - \log Df(a_i)|.$$

- For any continuous and $\mathbb{Z}$-periodic function ϕ, let:

$$|\phi|_0 = \|\phi\|_0 = \sup_{x \in \mathbb{R}} |\phi(x)|.$$

- Let $0 < \gamma' < 1$. The map $\phi \in C^0(\mathbb{T}^1)$ is Holder of order γ' if:

$$|\phi|_{\gamma'} = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^{\gamma'}} < +\infty.$$

  Let $\gamma \geq 1$ be a real number. All along the paper, we write $\gamma = r + \gamma'$ with $r \in \mathbb{N}$ and $0 \leq \gamma' < 1$.
- A function $\phi \in C^\gamma(\mathbb{T}^1)$ if $\phi \in C^r(\mathbb{T}^1)$ and if $D^r\phi \in C^{\gamma'}(\mathbb{T}^1)$. The set $C^\gamma(\mathbb{T}^1)$ is endowed with the norm:

$$\|\phi\|_\gamma = \max\left(\max_{0 \leq j \leq r} \|D^j \phi\|_0, |D^r \phi|_{\gamma'}\right).$$

  If $\gamma = 0$ or $\gamma \geq 1$, the C^γ-norm of ϕ is indifferently denoted $\|\phi\|_\gamma$ or $|\phi|_\gamma$. Thus, when possible, we favor the simpler notation $|\phi|_\gamma$.
- If $x \in \mathbb{T}^1$ and $\tilde{x}$ is a lift to $\mathbb{R}$, then:

$$|x| = \inf_{p \in \mathbb{Z}} |\tilde{x} + p|.$$

- For $x, y \in \mathbb{R}$, if $x \leq y$, $[x, y]$ denotes $\{t \in \mathbb{R}, x \leq t \leq y\}$ and if $x \geq y$, $[x, y]$ denotes $\{t \in \mathbb{R}, y \leq t \leq x\}$.
- For $\alpha \in \mathbb{R}$, we denote $R_\alpha \in D^\infty(\mathbb{T}^1)$ the map $x \mapsto x + \alpha$.
- An irrational number $\alpha \in DC(C_d, \beta)$ satisfies a Diophantine condition of order $\beta \geq 0$ and constant $C_d > 0$ if for any rational number p/q, we have:

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C_d}{q^{2+\beta}}.$$

Moreover, if $\beta = 0$, then α is of constant type C_d.

- Let $\alpha_{-2} = \alpha$, $\alpha_{-1} = 1$. For $n \geq 0$, we define a real number α_n (the *Gauss sequence* of α) and an integer $\hat{a}_n$ by the relations $0 < \alpha_n < \alpha_{n-1}$ and
$$\alpha_{n-2} = \hat{a}_n \alpha_{n-1} + \alpha_n.$$
- In the following statements, $C_i[a, b, \ldots]$ denotes a positive numerical function of real variables $a, b, \ldots$, with an explicit formula that we compute.
  $C[a, b, \ldots]$ denotes a numerical function of $a, b, \ldots$, with an explicit formula that we do not compute.
- We use the notations $a \wedge b = a^b$, $e^{(n)} \wedge x$ the n^{th}-iterate of $x \mapsto \exp x$, $\lfloor x \rfloor$ for the largest integer such that $\lfloor x \rfloor \leq x$, and $\lceil x \rceil$ for the smallest integer such that $\lceil x \rceil \geq x$.

We recall Yoccoz's theorem [14]:

THEOREM 1.1. — *Let $k \geq 3$ be an integer and $f \in D^k(\mathbb{T}^1)$. We suppose that the rotation number α of f is Diophantine of order β. If $k > 2\beta + 1$, there exists a diffeomorphism $h \in D^1(\mathbb{T}^1)$ conjugating f to R_α. Moreover, for any $\eta > 0$, h is of class $C^{k-1-\beta-\eta}$.*

1.2. Statement of the results

1.2.1. C^1 *estimations*

THEOREM 1.2. — *Let $f \in D^3(\mathbb{T}^1)$ be of rotation number α, such that α is of constant type C_d. Then there exists a diffeomorphism $h \in D^1(\mathbb{T}^1)$ conjugating f to R_α, which satisfies the estimation:*
$$|Dh|_0 \leq e \wedge \left(\frac{C_1[W(f), |Sf|_0]}{C_d} \right).$$
The expression of $C_{1.2}$ is given in page 681.

More generally, for a Diophantine rotation number $\alpha \in DC(C_d, \beta)$, we have:

THEOREM 1.3. — *Let $k \geq 3$ be an integer and $f \in D^k(\mathbb{T}^1)$. Let $\alpha \in DC(C_d, \beta)$ be the rotation number of f. If $k > 2\beta + 1$, then there exists a diffeomorphism $h \in D^1(\mathbb{T}^1)$ conjugating f to R_α, which satisfies the estimation:*
$$|Dh|_0 \leq C_2[k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}]. \tag{1}$$
The expression of C_2 is given in page 693.

Moreover, if $k \geq 3\beta + 9/2$, we have:
$$|Dh|_0 \leq e^{(3)} \wedge \left(C_3[\beta] C_4[C_d] C_5[|Df|_0, W(f), |Sf|_0] C_6[|Sf|_{\lceil 3\beta+3/2 \rceil}] \right). \tag{2}$$
The expressions of C_2, C_2, C_2, C_2 are given in page 695.