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SINGULAR PSEUDODIFFERENTIAL CALCULUS FOR WAVETRAINS AND PULSES

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ABSTRACT. — We generalize the analysis of [14] and develop a singular pseudodifferential calculus. The symbols that we consider do not satisfy the standard decay with respect to the frequency variables. While in [14], the remainders in the symbolic calculus were seen to be merely bounded operators on L^2, whose norm was measured with respect to some small parameter, we show here a smoothing property for the remainders. Due to a nonstandard decay in the frequency variables, the smoothing effect takes place in a scale of anisotropic, and singular, Sobolev spaces. Our analysis allows to extend the results of [14] on the existence of highly oscillatory solutions to nonlinear hyperbolic problems by dropping the compact support condition on the data. The results are also used in our companion works [7, 9] to justify nonlinear geometric optics with boundary amplification, which corresponds to a more singular regime than the one considered in [14]. The analysis is carried out with either an additional real

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or periodic variable in order to cover problems for pulses or wavetrains in nonlinear geometric optics.

Résumé (*Calcul pseudodifférentiel singulier pour les trains d'ondes et les pulses*)
Nous généralisons l'analyse de [14] et construisons un calcul pseudodifférentiel singulier pour des symboles ne vérifiant pas les hypothèses classiques de décroissance fréquentielle. Les résultats de [14] montraient que les restes du calcul symbolique étaient des opérateurs bornés sur L^2, dont la norme d'opérateur était contrôlée par rapport à un petit paramètre. Nous démontrons ici un effet régularisant pour ces restes dans une échelle d'espaces de Sobolev anisotropes. Notre analyse permet d'étendre les résultats de [14] sur l'existence de solutions hautement oscillantes de problèmes hyperboliques non-linéaires en s'affranchissant de l'hypothèse de support compact des données. Nos résultats sont aussi utilisés dans les articles compagnons [7, 9] pour justifier un régime d'optique géométrique non-linéaire avec amplification sur le bord. L'analyse est menée ici avec une variable rapide réelle ou bien périodique de manière à traiter des problèmes d'optique géométrique pour des pulses ou des trains d'ondes.

1. Introduction

Nonlinear geometric optics is devoted to the construction and the analysis of highly oscillatory solutions to some partial differential equations. In the context of hyperbolic partial differential equations, one of the main issues is to prove existence of a solution to the highly oscillatory problem on a time interval that is independent of the (small) wavelength. Such uniform existence results cannot follow from a naive application of a standard existence result in some functional space, say a Sobolev space H^s, because the sequence of initial and/or boundary data does not remain in a fixed ball of H^s. A now classical procedure for proving uniform existence results is to work on singular problems with additional variables and to prove uniform energy estimates with respect to the singular parameter. This strategy was used in [12] for the hyperbolic Cauchy problem and adapted in [14] to hyperbolic initial boundary value problems. Energy estimates in [14] are much more difficult to obtain than in [12] and are proved by using a singular pseudodifferential calculus[(1)]. The operators are pseudodifferential in the singular derivative $\partial_x + \beta \, \frac{\partial_\theta}{\varepsilon}$. The calculus of [14] is adapted to boundary value problems that satisfy a maximal energy estimate, that is an L^2 estimate with no loss derivative. In particular, remainders are bounded operators on L^2 whose norm is controlled with respect to some parameter γ. This parameter arises from a Laplace transform with respect to the

(1) The calculus is used partly to diagonalize the equations microlocally, and perform energy estimates on each coordinate. As detailed in the introduction of [14], symmetry arguments as in [12] are usually of no help in the study of boundary value problems.

time variable. Such terms of order 0 can be absorbed in the energy estimates by choosing γ large enough.

In [6], two of the authors have studied and justified geometric optics expansions with an amplification phenomenon for a certain class of *linear* hyperbolic boundary value problems. For linear problems, uniform existence is no source for concern. In the companion article [7], we extend the result of [6] to *semilinear* problems in the *weakly nonlinear* regime. Namely, we study solutions to the following boundary value problems

$$\begin{aligned} &\left(\partial_t + \sum_{j=1}^{d} A_j\, \partial_j\right) v_\varepsilon + D(\varepsilon\, v_\varepsilon)\, v_\varepsilon = 0 \qquad \text{in } \{x_d > 0\}, \\ &B(\varepsilon\, v_\varepsilon)\, v_\varepsilon = \varepsilon\, G\left(x', \frac{x' \cdot \beta}{\varepsilon}\right) \qquad \text{on } \{x_d = 0\}, \end{aligned}$$

where the source term G and the solution v_ε vanish for $t < 0$. One major issue in [7] is to prove that the amplification phenomenon exhibited in [6] combined with the nonlinearity of the zero order term $D(\varepsilon\, v_\varepsilon)\, v_\varepsilon$ does not rule out existence of a solution on a fixed time interval. Our strategy in [7] is to study a singular problem for which we need to prove uniform estimates. As in [6], the linearized problems in [7] satisfy a weak energy estimate with a loss of one tangential derivative.(2) Such estimates with a loss of derivative were originally proved in [5] and are optimal, as shown in [6]. The amplification of oscillations is more or less equivalent to the loss of derivatives in the estimates. Compared with [14], we now need to control our remainders by showing that they are smoothing operators, otherwise we can not absorb these errors in the energy estimates. Moreover, since the nonlinear problems of [7] are solved by a Nash-Moser procedure where we use smoothing operators (typically frequency cut-offs), it is crucial to extend all the results on the singular calculus of [14] by including the following features:

- The symbols should not be assumed to be independent of the space variables outside of a compact set. Otherwise, we would face a lot of difficulties with the smoothing procedure in the Nash-Moser iteration.
- The remainders in the calculus of [14] should be smoothing operators when they were merely bounded operators on L^2 with a small ($O(\gamma^{-1})$, γ large) norm in [14]. Moreover, we desire more systematic and easily applicable criteria than in [14] for determining the mapping properties of remainders.

As far as we checked, it seems that showing the smoothing property could be achieved with the techniques of [14]. However, these techniques heavily use

(2) More precisely, the loss in [7] is a loss of a singular derivative $\partial_x + \beta\, \frac{\partial_\theta}{\varepsilon}$.

the fact that the symbols are independent of the space variables outside of a compact set, and a major goal here is to get rid of this assumption. We thus adopt a different strategy that is based on a careful application of the Calderón-Vaillancourt Theorem. Our motivation for doing so is that our symbols lack the standard isotropic decay of pseudodifferential symbols, and this makes the classical proofs inapplicable. The situation is even worse because some results on adjoints or products of singular pseudodifferential operators seem not to hold. For instance, asymptotic expansions of symbols do not hold beyond the first term, and even the justification of the first term in the expansion depends on the order of the operators. Our final results are thus in some ways rather weak, but they seem to be more or less the best one can hope for in such a singular scaling. Fortunately, the calculus is strong enough to be applicable to a variety of geometric optics problems for both wavetrains and pulses, see, e.g., [7].

We thus review the results of [14] by improving them along the lines described above. For practical purposes, we have found it convenient to first prove general results on L^2-boundedness of pseudodifferential and oscillatory integral operators. The calculus rules are then more or less "basic" applications of the general results. We have also found it convenient to include in the same article, the results for both the whole space and the periodic framework. Results in the case of the whole space are used in [9] to deal with pulse-like solutions to hyperbolic boundary value problems, while the companion article [7] is devoted to wavetrains.

PART I

SINGULAR PSEUDODIFFERENTIAL CALCULUS FOR WAVETRAINS

2. Functional spaces

In all this article, functions may be valued in $\mathbb{C}$, $\mathbb{C}^N$ or even in the space of square matrices $\mathscr{M}_N(\mathbb{C})$ (or $\mathbb{C}^{N\times N}$). Products have to be understood in the sense of matrices when the dimensions agree. If $M \in \mathscr{M}_N(\mathbb{C})$, M^* denotes the conjugate transpose of M. The norm of a vector $x \in \mathbb{C}^N$ is $|x| := (x^*\,x)^{1/2}$. If x, y are two vectors in $\mathbb{C}^N$, we let $x \cdot y$ denote the quantity $\sum_j x_j\,y_j$, which coincides with the usual scalar product in $\mathbb{R}^N$ when x and y are real.