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CYCLE STRUCTURE OF THE INTERCHANGE PROCESS AND REPRESENTATION THEORY

by Nathanaël Berestycki & Gady Kozma

Abstract. — Consider the process of random transpositions on the complete graph K_n. We use representation theory to give an exact, simple formula for the expected number of cycles of size k at time t, in terms of an incomplete Beta function. Using this we show that the expected number of cycles of size k jumps from 0 to its equilibrium value, $1/k$, at the time where the giant component of the associated random graph first exceeds k. Consequently we deduce a new and simple proof of Schramm's theorem on random transpositions, that giant cycles emerge at the same time as the giant component in the random graph. We also calculate the "window" for this transition and find that it is quite thin. Finally, we give a new proof of a result by the first author and Durrett that the random transposition process exhibits a certain slowdown transition. The proof makes use of a recent formula for the character decomposition of the number of cycles of a given size in a permutation, and the Frobenius formula for the character ratios.

Résumé (*Structure des cycles dans le processus de transpositions et théorie des représentations*)

Nous considérons le processus de transpositions aléatoires sur le graphe complet K_n. Nous utilisons la théorie des représentations pour donner une formule exacte et simple pour l'espérance du nombre de cycles de taille k à un temps t, en termes d'une fonction Beta incomplète. À l'aide de cette formule nous montrons que cette

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quantité saute de 0 à sa valeur d'équilibre au précisément au moment où la composante géante du graphe aléatoire associé devient plus grande que k. Nous en déduisons une nouvelle preuve du résultat de Schramm sur les transpositions aléatoires, qui montre que les cycles géants apparaîssent au même moment que la composant géante dans le graphe aléatoire. Nous calculons également la fenêtre de cette transition, qui est particulièrement étroite. Finalement nous obtenons une preuve nouvelle d'un résultat du premier auteur et de Durrett sur la décélération du processus de transpositions. La preuve repose sur une formule récemment établie donnant la décomposition en caractères du nombre de cycles d'une taille donnée dans une permutation, ainsi que la formule de Frobenius pour les rapports de caractères.

1. Introduction and main results

Consider the complete graph K_n on n vertices, and let σ_t be the random walk on S_n that results when considering the interchange process on K_n. That is, σ_t is the usual random transposition process (see e.g. [17]) sped up by a factor $\binom{n}{2}$.

Our first result gives an exact and surprisingly simple formula for the expected number of cycles of size k at time t.

THEOREM 1. — *Fix any $1 \le k \le n$ and let $s_k(t)$ be the number of cycles of size k at time t in σ_t. Then*

$$\mathbb{E}(s_k(t)) = \binom{n}{k}\left[\frac{1}{k}x\phi(x) + \int_x^1 \phi(y)dy\right],$$

where $\phi(y) = y^{n-k}(1-y)^{k-1}$ and $x = e^{-tk}$.

This integral is known as the incomplete beta function (the integral from 0 to 1 is the regular beta function). The proof of Theorem 1 is based on representation theory. The key argument is a formula of Gil Alon and one of us [1] for the character decomposition of the number of cycles of a permutation, as well as Frobenius' formula for the values of the character ratios.

Our second result uses the above formula to show that in the limit as $n \to \infty$, the quantity $\mathbb{E}(s_k(t))$ exhibits a sharp transition from the value 0 to $1/k$ at a time $t_{n,k}$ which is essentially $(-1/k)\log(1-k/n)$, assuming that k is moderately big (of order at least $\sqrt{n}$). We note that this time is an order magnitude smaller than the mixing time for σ_t which (with this parametrization) is $(\log n)/n$ (see [9] or [4]). The width of this transition is shown to be order $1/n^{3/2}$ when k is of order n (which corresponds to a width of order $\sqrt{n}$ in the traditional scaling of random transpositions). This is reminiscent of the cutoff phenomenon for the mixing time, see [9], [15] or [14, §18] for the cutoff phenomenon.

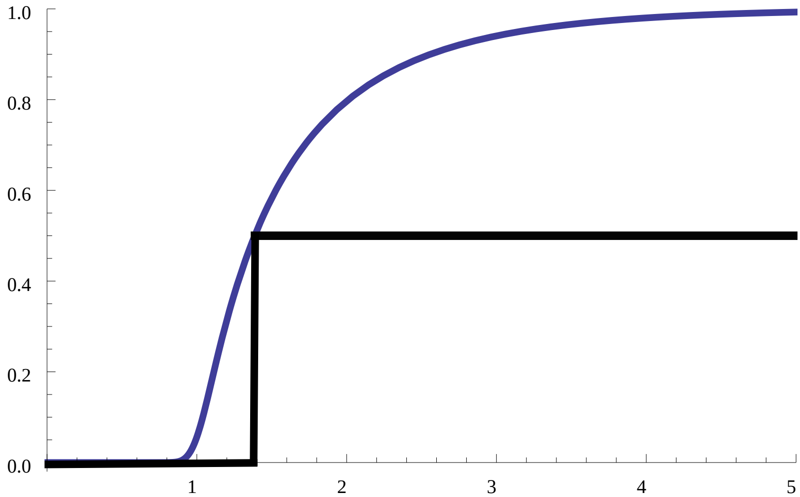

FIGURE 1. Approximate plots of $(n^2/k)\mathbb{E}(s_k(t/n))$, together with the relative size of the giant component, for $n = 200$ and $k = 100$. The scaling factor in front is chosen so that if $k/n \to \alpha$ then the limiting step function takes the values 0 and α.

THEOREM 2. — *Let $1 \le k < n$ and let $t_{n,k}$ be the unique t such that $e^{-kt} = (n-k)/(n-1)$. Then*

$$\left|\mathbb{E}(s_k(t)) - \frac{1}{k}\mathbf{1}_{\{t>t_{n,k}\}}\right| \le Cq\exp\left\{-c(n-k)\min\{|t-t_{n,k}|^2k^2, 1\}\right\}$$

where $q = q_n(k)$ is a polynomial factor, $q = n^{3/2}k^{-3/2}(n-k)^{-1/2}$.

(Here and below c and C pertain to unspecified positive universal constants, possibly different from one place to another).

At first sight it might seem as if Theorem 2 cannot possibly hold. After all, a large cycle of σ_t (say of size $\frac{1}{3}n$) is necessarily contained in the giant component of a corresponding Erdős-Rényi graph $G(n,t)$ with edge density t (see e.g. [17]). It is easy to check that $t_{n,k}$ is the first time that the giant component has a relative size which exceeds 1/3, so it is clear that $\mathbb{E}(s_k(t))$ must be close to 0 before $t_{n,k}$. What Theorem 2 says is that $\mathbb{E}(s_k(t))$ suddenly reaches its equilibrium value precisely at that time, and does not change afterward. Note however that as t increases above $t_{n,k}$, the cluster continues to grow. (See Figure 1). So how come the probability for a cycle of size exactly $\frac{1}{3}n$ does not grow with it? The answer lies in examining a completely random permutation. The expected number of cycles of length k in a random permutation of n elements