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INSTABILITY OF TYPE II BLOW UP FOR THE QUINTIC NONLINEAR WAVE EQUATION ON \mathbb{R}^{3+1}

BY JOACHIM KRIEGER & JOULES NAHAS

ABSTRACT. — We prove that the blow up solutions of type II character constructed by Krieger-Schlag-Tataru [10] as well as Krieger-Schlag [9] are unstable in the energy topology in that there exist open data sets whose closure contains the data of the preceding type II solutions and such that data in these sets lead to solutions scattering to zero at time $t = +\infty$.

RÉSUMÉ (*Instabilité d'explosion de type II pour l'équation des ondes non-linéaire de degré 5 sur \mathbb{R}^{3+1}*)

On montre que les solutions explosives de type II construites par Krieger-Schlag-Tataru [9] sont instables dans l'espace d'énergie au sens qu'il existe des ensembles ouverts de données initiales dont la fermeture contient les données initiales des solutions de type II et telles que les solutions correspondantes existent globalement en temps positif et s'évanouissent vers $t = +\infty$.

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1. Introduction

We consider the quintic focussing wave equation on \mathbb{R}^{3+1} , of the form

$$(1.1) \quad \square u = u^5, \quad \square = \partial_t^2 - \Delta$$

in the radial context, i. e. $u(t, x) = v(t, |x|)$. This equation is of energy critical and focussing type and serves as a convenient model for more complicated energy critical models, such as Wave Maps in $2 + 1$ dimensions with positively curved targets, or Yang-Mills equations in $4 + 1$ dimensions as well as related problems of Schrödinger type. In fact, for example recent progress on (1.1) in [1] has led to analogous progress for the energy critical focussing NLS in $3 + 1$ dimensions, [12]. The focussing character of (1.1) leads to finite time blow up, which is most easily manifested by the explicit solutions of ODE type

$$u(t, x) = \frac{\left(\frac{3}{4}\right)^{\frac{1}{4}}}{(T-t)^{\frac{1}{2}}}$$

for arbitrary T . Truncating the data of these solutions at time $t = 0$ to force finiteness of $\int_{\mathbb{R}^3} [u_t^2(0, \cdot) + |\nabla_x u(0, \cdot)|^2] dx$, one easily verifies that

$$\lim_{t \rightarrow T} \int_{\mathbb{R}^3} [u_t^2(t, \cdot) + |\nabla_x u(t, \cdot)|^2] dx = +\infty$$

One says the blow up is of type I. By contrast, a finite time blow up solution with

$$\limsup_{t \rightarrow T} \int_{\mathbb{R}^3} [u_t^2(t, \cdot) + |\nabla_x u(t, \cdot)|^2] dx < +\infty$$

where T is the blow up time is called of type II. Existence of the latter type of solution for (1.1) is rather subtle and appears to have first been accomplished in [10], see also [9], and Hillairet-Raphaël's paper [5] for more stable blow up solutions in the $4 + 1$ -dimensional context. The works [10], [9] show that denoting

$$u_\lambda(t, x) := \lambda^{\frac{1}{2}} u(\lambda t, \lambda x), \quad \lambda > 0$$

problem (1.1) admits type II blow up solutions (of energy class) of the form

$$(1.2) \quad u(t, x) = W_{\lambda(t)}(x) + \varepsilon(t, x), \quad \lambda(t) = (-t)^{-1-\nu}, \quad t \in [-t_0, 0), \quad t_0 \lesssim 1,$$

for $\nu > 0$, with $W(x)$ denoting the ground state static solution

$$W(x) = \frac{1}{\left(1 + \frac{|x|^2}{3}\right)^{\frac{1}{2}}}$$

More precisely, the solutions constructed in [10], [9] admit a precise description of the radiation term $\varepsilon(t, x)$ inside the light cone $\{r \leq |t|\}$ of the form

$$\varepsilon(t, x) = O\left(\lambda^{\frac{1}{2}}(t) \frac{R}{(\lambda(t)|t|)^2}\right), \quad R = \lambda(t)|x|,$$

and furthermore $\varepsilon(t, \cdot) \in \dot{H}^1$ with

$$\|(\varepsilon, \varepsilon_t)\|_{(\dot{H}^1 \times L^2)(|x| \leq t)} \lesssim (\lambda(t)|t|)^{-\frac{1}{2}}$$

By contrast, outside of the light cone, we can only assert that

$$\|\nabla_{t,x}\varepsilon\|_{L^2(|x| \geq t)} \leq \delta_*$$

where we may arrange for δ_* to be arbitrarily small. Indeed, this is consistent with the fact proved in [3] that type II blow up solutions must have energy strictly larger than that of the ground state.

We also mention that analogous *infinite time blow up solutions* were constructed in [1]. See also [5] for type II blow up with a different rate for the energy critical NLW in 4 + 1 dimensions.

The remarkable series of papers [3] - [4] recently gave a complete classification of the possible type II solutions, on finite or infinite time intervals, in the radial context for (1.1). These works show that any type II solution decouples as a sum of dynamically rescaled ground states $\pm W$ at diverging scales, plus an error that remains regular at blow up time (or radiates to zero in the infinite time case). In these works, it is intimated that all such type II solutions ought to be unstable in the energy topology, and in fact ought to constitute the boundary of both the set of solutions existing globally and scattering to zero, as well as those blowing up of type I. Indeed, it is only the latter two which are readily observable in numerical experiments.

The recent work [7] gives a rather precise description of the instability of the static solution W with respect to a suitably strong topology.

Here, we show that the solutions constructed in [10], [9] are unstable in the energy topology, provided ε has sufficiently small energy. Specifically, we have

THEOREM 1.1. — *There exists $\delta_* > 0$ with the following property: let $u(t, x)$ be one of the type II blow up solutions constructed in [10], [9]*

$$(1.3) \quad u(t, x) = W_{\lambda(t)}(x) + \varepsilon(t, x), \quad \lambda(t) = (-t)^{-1-\nu}$$

satisfying the a priori condition

$$\limsup_{t \in [-t_0, 0)} \|\nabla_{t,x}\varepsilon(t, \cdot)\|_{L_x^2} < \delta_*, \quad t_0 \lesssim 1.$$

Then there exists an open set of data U in the energy topology at time $-t_0$, with

$$(u(-t_0, \cdot), u_t(-t_0, \cdot)) \in \overline{U},$$

and such that all data in U lead to solutions existing globally and scattering to zero in forward time. Also, there is an open set of data V with

$$(u(-t_0, \cdot), u_t(-t_0, \cdot)) \in \overline{V},$$