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FACTORS OF PISOT TILING SPACES AND THE COINCIDENCE RANK CONJECTURE

by Marcy Barge

Abstract. — We consider the structure of Pisot substitution tiling spaces, in particular, the structure of those spaces for which the translation action does not have pure discrete spectrum. Such a space is always a measurable m-to-one cover of an action by translation on a group called the maximal equicontinuous factor. The integer m is the coincidence rank of the substitution and equals one if and only if translation on the tiling space has pure discrete spectrum. By considering factors intermediate between a tiling space and its maximal equicontinuous factor, we establish a lower bound on the cohomology of a one-dimensional Pisot substitution tiling space with coincidence rank two and dilation of odd norm. The Coincidence Rank Conjecture, for coincidence rank two, is a corollary.

Résumé (*Facteurs de l'espace des pavages d'une substitution de Pisot et conjecture du rang de coïncidence*)

Nous considérons la structure de l'espace des pavages d'une substitution de Pisot, en particulier dans le cas où l'action par les translations n'a pas de spectre purement discret. Un tel espace est toujours recouvrement presque partout de degré m d'une translation sur un groupe. Ce groupe s'appelle le facteur maximal équicontinu. L'entier m est le rang de coïncidence de la substitution et il vaut 1 si et seulement si l'action par les translations a un spectre purement discret. En tenant compte des facteurs intermédiaires entre l'espace de pavage et son facteur maximal équicontinu, nous établissons une borne inférieure sur la cohomologie des espaces des pavages d'une substitution de Pisot unidimensionelle avec rang de coïncidence 2 et dilatation de norme impaire. La conjecture du rang de coïncidence, pour un rang de coïncidence égal a 2, en découle en tant que corollaire.

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1. Introduction

A subset $\mathcal{D}$ of $\mathbb{R}^n$ is a *Delone set* if it is relatively dense and uniformly discrete. Such a set has *finite local complexity* (FLC) if, for each $r > 0$, there are, up to translation, only finitely many subsets of $\mathcal{D}$ of diameter less than r and is *repetitive* if, for each r there is an R so that every Euclidean ball of radius R intersected with $\mathcal{D}$ contains a translated copy of every subset of $\mathcal{D}$ of diameter less than r. Repetitive FLC Delone sets (RFLC) are commonly employed as models of the atomic structure of materials and there is a well-developed diffraction theory to go with these models ([16], [4], [18], [22]). Of course if a Delone set is a periodic lattice, then its diffraction spectrum is pure point. But there is also a vast menagerie of non-periodic repetitive FLC Delone sets with pure point diffraction spectra (the so-called quasicrystals) and it is a very subtle problem to predict which sets will have this property.

An effective method of encoding the structure of a single Delone set $\mathcal{D}$ is to form the space, called the *hull* of $\mathcal{D}$ and denoted $\Omega_{\mathcal{D}}$, consisting of all Delone sets that are, up to translation, locally indistinguishable from $\mathcal{D}$. The hull has a metric topology in which two Delone sets are close if a small translate of one is identical to the other in a large neighborhood of the origin. If $\mathcal{D}$ is RFLC then $\Omega_{\mathcal{D}}$ is compact and connected and $\mathbb{R}^n$ acts minimally on $\Omega_{\mathcal{D}}$ by translation. A highlight of this approach is that (in the substitutive context considered here) $\mathcal{D}$ has pure point diffraction spectrum if and only if the $\mathbb{R}^n$-action on $\Omega_{\mathcal{D}}$ has pure discrete dynamical spectrum (for the backwards direction see [15], for the equivalence see [20], and for a prototypical result in the symbolic context see [24]).

In this article we consider Delone sets that arise from a substitution rule. In this case, the eigenfunctions of the $\mathbb{R}^n$-action on $\Omega_{\mathcal{D}}$ can all be taken continuous ([26]) and then collectively determine a continuous map g factoring the $\mathbb{R}^n$ action on $\Omega_{\mathcal{D}}$ to a translation action on a compact abelian group. The map g is called the *maximal equicontinuous factor map* and it turns out that the $\mathbb{R}^n$-action on $\Omega_{\mathcal{D}}$ has pure discrete spectrum if and only if g is a.e. one-to-one ([10]).

Two ingredients make up a substitution: a linear expansion $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$; and a subdivision rule. For the $\mathbb{R}^n$-action on $\Omega_{\mathcal{D}}$ to have any discrete component in its spectrum, the eigenvalues of Λ must satisfy a a *Pisot family* condition ([21]). In particular, if Λ is simply a dilation by the number $\lambda > 1$ (the "self-similar" case), the $\mathbb{R}^n$-action will have a discrete component if and only if λ is a Pisot number. (Recall that a Pisot number is an algebraic integer, all of whose algebraic conjugates lie strictly inside the unit circle.) For the purpose of determining which substitution Delone sets $\mathcal{D}$ have pure point diffraction spectrum, we are thus led to consider factors of the spaces $\Omega_{\mathcal{D}}$ for which the expansion is Pisot family.

For convenience, we will consider substitution tilings rather than substitution Delone sets. (These notions are essentially equivalent - see [20] and [19].) A *tile* is a compact, topologically regular subset of $\mathbb{R}^n$ (perhaps "marked"); for a given substitution, there are only finitely many translation equivalence classes of tiles and each such class is a *type*. A substitution expands tiles by a linear map Λ, then replaces the expanded tile by a collection of tiles. The *substitution matrix* is the $d \times d$ matrix M (d being the number of distinct tile types) whose $ij-$th entry is the number of tiles of type i replacing an inflated tile of type j and the *tiling space* associated with a substitution Φ is the collection Ω_Φ of all tilings T of $\mathbb{R}^n$ with the property that each finite patch of T occurs as a sub patch of some repeatedly inflated and substituted tile. To pass from a tiling T to a Delone set $\mathcal{D}$, simply pick a point from each tile in T: if the points picked are "control points" (see [21]), $\mathcal{D}$ will be a substitution Delone set with linear expansion Λ. For a thorough treatment of the basics of substitution tiling spaces, see [2].

A 1-dimensional Pisot substitution is *irreducible* if the characteristic polynomial of its substitution matrix is irreducible over $\mathbb{Q}$, and is *unimodular* if that matrix has determinant ± 1. The following has become known as the *Pisot Substitution Conjecture*:

CONJECTURE 1. — *(PSC) If Φ is a 1-dimensional irreducible, unimodular, Pisot substitution then the $\mathbb{R}$-action on Ω_Φ has pure discrete spectrum.*

The dimension of the first (Čech , with rational coefficients) cohomology of a 1-dimensional substitution tiling space is at least as large as the degree of the associated dilation. Replacing irreducibility in the PSC by minimality of cohomology results in the *Homological Pisot Conjecture*:

CONJECTURE 2. — *(HPC) If Φ is a 1-dimensional substitution with dilation a Pisot unit of degree d and the first Čech cohomology $H^1(\Omega_\Phi)$ has dimension d, then the $\mathbb{R}$-action on Ω_Φ has pure discrete spectrum.*

As noted above for substitution Delone sets, the $\mathbb{R}^n$-action on Ω_Φ has pure discrete spectrum if and only if the maximal equicontinuous factor map g is a.e. one-to-one. It is proved in [10] that, for Pisot family Φ, there is $cr(\Phi) \in \mathbb{N}$ (called the *coincidence rank* of Φ) so that g is a.e. $cr(\Phi)$-to-one. The following *Coincidence Rank Conjecture* (see [5]) extends the HPC to the non-unit case.

CONJECTURE 3. — *(CRC) If Φ is a 1-dimensional substitution with Pisot dilation λ and the dimension of $H^1(\Omega_\Phi)$ equals the degree of λ, then $cr(\Phi)$ divides the norm of λ.*